J. Condensed Matter Nucl. Sci. 11 (2013) 101–122

Research Article

Compatibility of Hydrino States and Quantum Mechanics

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Abstract

Rathke's assertion [*New J. Phys.* **7** (2005) 127] that states with binding energy and size below those of known literature values are incompatible with quantum mechanics is corrected by reviewing the analytically known Coulomb solution of the Klein–Gordon equation with binding energy of order mc^2 and size of order of the Compton wavelength. This is an example of a quantum state, which is mathematically acceptable in the sense of being square integrable and having a finite binding energy but yet is rejected as unphysical due in part to the point-nucleus nature of the model. Then the Dirac equation is studied for the existence of states which are similarly mathematically acceptable but whose physical acceptability requires physical judgment. States of Landau symmetry are found which meet these criteria. The existence of states of ambiguous physical interpretation for both the Klein–Gordon and Dirac equations depends on using a point-nucleus versus a finite-nucleus potential model. On using a realistic model for the charge distribution of the proton, a Klein–Gordon state is found in the binding range of 5 keV, but no state is found for the Dirac equation.
 ISSN 2227-3123

Keywords: Dirac equation, Hydrino states, Klein–Gordon equation

1. Introduction

Rathke [1] has presented a critical analysis of the hydrino model [2,3], which has been invoked to interpret experimental results which have appeared in respectable physics journals [4–7]. He also reviews the incompatibility of hydrino states with quantum mechanics. It is this last area which is the focus of the present paper. What I mean by a hydrino state, in a generic sense, is any quantum state with binding in a Coulomb potential characterized by a binding energy and size below those of the known states of the Schrödinger, Klein–Gordon, or Dirac equations. This point has arisen historically whenever experimental results appear to be uninterpretable using standard theory. The earliest example following the discovery of the neutron was likely the proposal that the neutron might be a small hydrogen atom. Margenau examined this question and concluded in his 1934 paper [8] that the Schrödinger or Dirac equations could not support such states unless the potential is modified at small distances from the origin. As a more recent example Evans [9] has studied an unconventional form of Dirac theory, called 4-space Dirac theory, in which the three spatial variables and the scaled time, ct, are treated on an equal footing to avoid any suggestion of a preferred reference frame. Evans' motivation was to find possible theoretical support for low-energy nuclear reactions (LENR), a field which is known pejoratively as

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 ISSN 2227-3123

"cold fusion." Hence, this subject is regarded as closed by many theorists, and Rathke's blanket assertion that hydrino states are incompatible with quantum mechanics, by which he seems to mean nonrelativistic quantum mechanics (although "quantum mechanics" certainly includes relativistic quantum mechanics), follows in this tradition. In any event the Rathke-hydrino controversy provides a sound opportunity to clarify much of the confusion which exists in the literature on this subject.

For example there is at least one known exception to Rathke's conclusion which, although it appeared in the third edition of Schiff's text on quantum mechanics [10], seems to be little known in the physics community. The Klein–Gordon equation has an analytically known state with binding energy of order mc^2 and Compton wavelength size. It is this state which is irregular at the origin in Schroedinger theory (or more precisely in the Schroedinger limit of Klein–Gordon theory obtained by neglecting terms scaling as α^2, where $\alpha = e^2/\hbar c$ is fine structure constant) and which is therefore unambiguously rejected as unphysical. In Klein–Gordon theory, however, the state is mathematically acceptable in the sense of being square integrable and having a finite binding energy. The state is rejected as unphysical by Schiff for two reasons. First he points out that the particle described by the Klein–Gordon equation has no spin and therefore cannot be an electron. The particle described by Schrödinger's equation also has no spin and yet its electronic properties in the nonrelativistic regime are described with stunning success. To be fair the Klein–Gordon equation fails to account for the observed magnetic fine structure of the atom, which depends on the electron's spin. Second Schiff rejects the Klein–Gordon state as unphysical because it is calculated using a point-nucleus model and therefore fails to account for the finite size of the proton. Nevertheless the Klein–Gordon point-nucleus result illustrates a principle of binding which is unknown in standard quantum theory, namely that an electromagnetic potential, even one of unit strength, can support binding with binding energy of order mc^2 without any modification of the potential at small distances from the center of attraction, as suggested by Margenau [8]. This principle appears to be unrecognized in the physics community, probably owing to arguments that the state is not physically realizable and that the Klein–Gordon equation is inapplicable to the electron. The principle depends on the nature of the relativistic motion and not on the strength of the potential, the latter of which derives purely from Schrödinger theory and is the basis for the wisdom, in beta decay for example, that an electron cannot reside initially inside the small volume of a nucleus because the uncertainty principle for momentum and position, $\Delta p\,\Delta r \geq \hbar$, would be violated for a Δr of nuclear size unless a potential of sufficient strength exists to produce a very largeΔp.

This proof-of-principle binding by the Klein–Gordon equation is reviewed in Section 2. Then in Section 3 the Dirac equation is examined from a new perspective. The general acceptability of Dirac's equation as the equation of motion for the electron rests fundamentally on its success in accounting for the observed magnetic fine structure of the atom and for the anomalous Zeeman effect, which was first successfully described by Pauli's equation, which is an *ad hoc* modification of Schrödinger's equation to account for the observed splitting of a *zero orbital angular momentum* state in the presence of a magnetic field. Pauli's spin vector, $\vec{\sigma}$, in Pauli's or in Dirac's equations occurs diagonally along the z-axis and is therefore compatible with the orbital angular momentum operator, $\vec{\ell}$, which is also is an Eigen operator along the z-axis. This magnetic-axis preference in Dirac's equation suggests that it may be possible for states having Landau symmetry to exist even in absence of a magnetic field. (Landau states [11,12] are states which are bound transversely to a uniform magnetic field along the quantization- or z-axis.) A suggestion that Landau states may exist is found in the non-diagonal nature of all four components of Dirac's wave function simultaneously when each of the large and small components (more in Section 3) is written in the standard way as a product of a radial function, the large or small component, and a two-component spinor, $\chi_{\kappa\mu}(\theta, \phi)$, where κ and μ are the "good"quantum numbers in the $\vec{j} = \vec{\ell} + \frac{1}{2}\vec{\sigma}$ or total angular-momentum representation. The spinors belonging to the large and small components have equal and opposite values of the κ quantum number. The series expansion in which all four components of Dirac's vector wave function can be brought into diagonal form comprises basis members having the Laudau symmetry, as we discuss in Section 3.

In view of the emergence of experiments by Mills and others whose interpretation suggests the existence of quantum

states, which are unknown in the literature, it is imperative to investigate quantum states whose elimination as unphysical on the application of boundary conditions may be ambiguous and may rest on physical judgment alone. In this paper I study the Dirac equation from this viewpoint. It is critical, as pointed out by Rathke, to use a Lorentz-invariant theory. Schrödinger theory is not Lorentz invariant and will yield nothing further on this subject. The Klein–Gordon equation is Lorentz invariant, but it is inapplicable to the electron. The Dirac equation is Lorentz invariant, and it is applicable to the electron.

Finally, I wish to emphasize that the approach in this paper is not based on the concepts or methodology of quantum electrodynamics (QED).

2. Klein–Gordon Equation. Regularization of Schrödinger Irregular States

Readers should recall that a criterion for an acceptable relativistic quantum EOM is that it is invariant to a Lorentz transformation. The scalar product of 4-vectors is always Lorentz invariant [13], and the Klein–Gordon equation follows from the scalar product of the covariant and contra variant 4-momentum,

$$(\gamma mc,\ -\gamma m\overrightarrow{v}) = \left(\frac{E}{c} - \frac{V}{c},\ i\hbar\overrightarrow{\nabla} + \frac{e}{c}\overrightarrow{A}\right),$$

operating on a scalar wave function equal to the Lorentz-constant *mc* times the wave function,

$$\left(\frac{1}{c}(E-V),\ i\hbar\vec{\nabla} + \frac{e}{c}\vec{A}\right)\cdot\left(\frac{1}{c}(E-V), -i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right)\psi = \left[\frac{1}{c^2}(E-V)^2 - (i\hbar\vec{\nabla} + \frac{e}{c}\vec{A})^2\right]\psi = mc\psi, \qquad (1)$$

where

$$E = \gamma mc^2 + V, \quad \gamma m\overrightarrow{v} = \overrightarrow{p} = \overrightarrow{P} - \frac{e}{c}\overrightarrow{A}, \quad \overrightarrow{P} \to i\hbar\vec{\nabla}$$

have been used where γ is the Lorentz factor and $\vec{P}$ is the generalized or canonical momentum. Let us look at the small-r equation for $V = -Ze^2/r$ and $\vec{A} = 0$, restricting ourselves to zero-angular-momentum states and keeping only dominant terms,

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\beta^2}{r^2}\right)g = 0, \qquad (2)$$

for $\beta = Ze^2/\hbar c$. Using $g = r^s$ in Eq. (2) the indicial equation for s is found by setting the coefficient of r^{s-2} equal to zero, $s(s-1) + 2s + \beta^2 = 0$, whose solutions are given by

$$s = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4\beta^2} \simeq -\beta^2,\ -1 + \beta^2$$

for the upper and lower signs, respectively. The upper sign is retained in the standard set of solutions in the literature. Notice that $\beta = 0$ for the Schrödinger equation such that the lower-sign solution can be unambiguously rejected as irregular at the origin and therefore as unphysical. (Although the irregular Schrödinger solution is square integrable and therefore normalizable, its expectation value of V diverges logarithmically such that the second criterion of an acceptable solution, a finite binding energy, is not satisfied.) Proceeding heuristically I propose a variational trial

solution over all r having the form, $\psi = Nr^{\beta^2-1}\mathrm{e}^{-wr}$, where the normalization constant is given by

$$N = \left(\int_0^\infty \mathrm{d}r\, r^2 r^{2\beta^2-2}\mathrm{e}^{-2wr}\right)^{-1/2} \cong (2w)^{1/2}, \tag{3}$$

and where w is a parameter chosen to minimize the energy, which is found from Eq. (1),

$$E^2 + 2EZe^2\langle\psi|o|\psi\rangle - m^2c^4 + \hbar^2c^2\left\{w^2 - w\left[2(\beta^2-1)+2\right]\langle\psi|o|\psi\rangle\right\} = 0, \tag{4}$$

where

$$\langle\psi|o|\psi\rangle = 2w\int_0^\infty \mathrm{d}r\, r^{2\beta^2-1}\mathrm{e}^{-2wr} = \frac{wr^{2\beta^2}\mathrm{e}^{-wr}}{\beta^2}|_0^\infty + \frac{2w^2}{\beta^2}\int_0^\infty \mathrm{d}r\, r^{2\beta^2}\mathrm{e}^{-2wr} \simeq \frac{w}{\beta^2}. \tag{5}$$

The approximately equals sign is used whenever $r^{2\beta^2} \simeq 1$, which it cannot be in the first term on the right-hand side of Eq. (5) after the second equality sign, since the limit of $r = 0$ is taken for finite β, but which it can be in the second term and in Eq. (3) since the contribution of $r^{2\beta^2}$ to the integral is negligible. Notice the cancellation of terms in the curly bracket in Eq. (4), including the cancellation of β^2. This is responsible for the binding since otherwise the kinetic energy would exceed the potential energy making binding impossible.

The energy is given by the quadratic-root formula,

$$E = -\frac{wZe^2}{\beta^2} \pm \sqrt{\left(\frac{wZe^2}{\beta^2}\right)^2 + m^2c^4 + \hbar^2c^2w^2}, \tag{6}$$

where the upper sign is taken to be the physical root. Minimizing E with respect to w, $w \simeq mc/\hbar$ and $E \simeq mcZe^2/\hbar$, which is identical to the leading term of the exact analytic energy given by Schiff's formula 51.16 [10],

$$E = mc^2\left(1 + \frac{\beta^2}{\lambda^2}\right)^{-1/2} = mc^2\beta(1+\beta^2)^{-1/2} \simeq \frac{mcZe^2}{\hbar}$$

using his formula 51.17 for $\lambda = s + 1 \simeq \beta^2$. Equation (6) is plotted in Fig. 1 versus w.

The binding energy is given by $E_\mathrm{b} = mc^2 - E$. Notice that Schiff asserts that s is the nonnegative solution of the indicial equation, his formula 51.18, in anticipation of his argument, which follows immediately, that when the finite size of the proton is considered then negative values of s are ruled out in the limit of a point-Coulomb source for a *finite* wave function at $r = 0$. But a finite wave function calculated for a finite-source model will of course have the limit of the positive-s point-source solution when the radius of the finite-source model is taken to zero. Schiff's exercise does not really lift the ambiguity of the positive-s, negative-s, point-source solutions in the sense that both solutions are normalizable with finite binding energies. Point Coulomb sources really do exist in nature, an electron–positron pair for example.

As pointed out earlier, we are concerned in this study with point center-of-force solutions which satisfy two criteria – normalizable with finite binding energy – in order to understand binding of order mc^2 which is due not to the strength of the potential, as in Schrödinger theory, but to the relativistic nature of the motion. Although the probability density is infinite as $r^{2\beta^2-2}$ at $r = 0$, the radial distribution is nevertheless finite. Notice that the size of the negative-s solution is comparable to the Compton wavelength and thus is still much larger than the size of the proton.

Finally, we comment on the double energy-root nature of the Klein–Gordon equation. This also occurs of course in Dirac's equation, and Dirac lifted the ambiguity by filling up the negative-energy levels with electrons such that

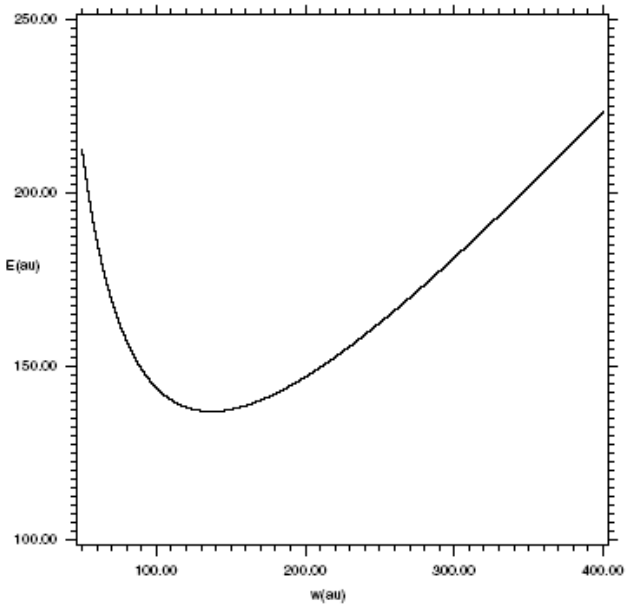

Figure 1. Energy versus variational parameter, w, for the negative-s state of the Klein–Gordon equation. The binding energy is $mc^2 - E$.

positive-energy electrons are forbidden from occupying negative-energy states, which averts the instability of the atom against spontaneous radiative transitions from positive-energy to negative-energy states. The Klein–Gordon equation is criticized on the basis that Dirac's interpretation of the negative-energy states for his own equation would not apply to the Klein–Gordon equation due to the absence of spin such that Pauli's exclusion principle is not obeyed. But the same criticism can be applied to Schrödinger's equation with respect to the positive-energy states. The shell structure of the atom depends entirely on an *ad hoc* antisymmetrization of a product of Schrödinger orbitals augmented by up or down spin states in order to satisfy the Pauli principle. This procedure is totally phenomenological and does not identify spin as a relativistic property of the electron, even though it derives in an *ab initio* sense from Dirac's equation, or explain how the spin of an individual electron is responsible in a causal sense for Fermi–Dirac statistics for an aggregate of many electrons.

Finally, having just mentioned the role of the negative-energy states and Dirac's interpretation of them as antimatter states, I will henceforth limit the investigations of this paper to positive-energy states, as in the Klein–Gordon example above.

3. Dirac Equation

Pauli modified Schrödinger's equation to account for the observed anomalous Zeeman effect, which is the splitting of states of zero orbital angular momentum ($\ell = 0$) in a magnetic field. He introduced the spin vector $\vec{\sigma}$ in an *ad hoc* sense, whose properties were such that

$$\frac{1}{2m}\left[\vec{\sigma}\cdot(i\hbar\vec{\nabla}+\frac{e}{c}\vec{A})\right]^2 = -\frac{\hbar^2}{2m}\nabla^2 + \frac{ie\hbar}{mc}\vec{A}\cdot\vec{\nabla} + \frac{e^2}{2mc^2}A^2 - \frac{e\hbar}{2mc}\vec{\sigma}\cdot(\vec{\nabla}\times\vec{A}), \tag{7}$$

where

$$\overrightarrow{\sigma} = \hat{\imath} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \hat{\jmath} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \hat{k} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{8}$$

and the vector identity,

$$(\vec{\sigma} \cdot \vec{C})(\vec{\sigma} \cdot \vec{D}) = \vec{C} \cdot \vec{D} + i\vec{\sigma} \cdot (\vec{C} \times \vec{D}), \tag{9}$$

has been used. In a spatially uniform magnetic field, $\vec{H}$,$\vec{A} = (1/2)\vec{H} \times \vec{r}$ such that the last three terms on the right-hand side of Eq. (7) are

$$V_m = -\frac{e\hbar}{2mc}(\ell_z + \sigma_z)H_z + \frac{e^2}{8mc^2}\rho^2 H_z^2 \tag{10}$$

for a field in the z-direction, where $\rho = \sqrt{x^2 + y^2}$. Notice that even in absence of V, the Coulomb potential, an electron can be bound harmonically in the transverse direction to z, in which direction the electron is free but whose states are split into magnetic sublevels. These are Landau states [11,12]. This point will be revisited when we analyze Dirac's equation for the possibility of the existence of states of Landau symmetry in the presence of V but absence of H_z owing to the diagonal nature of $\vec{\sigma}$ along the z-axis, as given by Eq. (8).

Dirac's equation is usually written as the Lorentz invariant found from the scalar product of the Dirac 4-γ matrix, $(\gamma_0, \vec{\gamma})$, and the electron's 4-momentum operating on a 4-component wave function to give the Lorentz constant mc times the wave function, thus,

$$\left[(\gamma_0, \vec{\gamma}) \cdot \left(i\hbar\frac{\partial}{c\partial t} - \frac{V}{c}, i\hbar\vec{\nabla} + \frac{e}{c}\vec{A}\right)\right]\psi_{\mathrm{D}} = \left[\gamma_0\left(i\hbar\frac{\partial}{c\partial t} - \frac{V}{c}\right) + \vec{\gamma} \cdot \left(i\hbar\vec{\nabla} + \frac{e}{c}\vec{A}\right)\right]\psi_{\mathrm{D}} = mc\psi_{\mathrm{D}}, \tag{11}$$

where the time-dependent operator form of E, $E \to i\hbar(\partial/\partial t)$, has been used and where $\gamma_0 = \beta$ and $\vec{\gamma} = \beta\vec{\alpha}$ on using Dirac's own $\vec{\alpha}$ and β matrices from his original derivation [14],

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \overrightarrow{\alpha} = \begin{pmatrix} 0 & \overrightarrow{\sigma} \\ \overrightarrow{\sigma} & 0 \end{pmatrix}, \tag{12}$$

where I is the identity matrix. The γ matrices form a Clifford algebra [15], which has been in the mathematics literature for some time. A separate step is then required to prove the Lorentz invariance of the wave equation itself [14].

Recalling that the scalar product of 4-vectors is always Lorentz invariant, Dirac's equation can be derived by further elucidating the close relationship between Dirac's equation and the spinorial form of Maxwell's equation, which has been studied continuously since 1928 [16–19]. Dirac's equation can be inferred from the scalar product of the electron's 4-momentum and an electromagnetic 4-potential, $(\Psi, \vec{\mathrm{X}})$, posited for the electron as follows,

$$\left(i\hbar\frac{\partial}{c\partial t} - \frac{V}{c}, i\hbar\vec{\nabla} + \frac{e}{c}\vec{A}\right)(\Psi, \overrightarrow{X}) = \left(i\hbar\frac{\partial}{c\partial t} - \frac{V}{c}\right)\Psi + \left(i\hbar\vec{\nabla} + \frac{e}{c}\vec{A}\right) \cdot \overrightarrow{X} = 0. \tag{13}$$

Using a carrier-wave expansions for $(\Psi, \vec{\mathrm{X}})$ in order to isolate a dominant frequency component of the 4-potential we obtain,

$$\Psi = \Psi_-(\vec{r}, t)\mathrm{e}^{-i\omega t} + \Psi_+(\vec{r}, t)\,\mathrm{e}^{i\omega t}, \tag{14}$$

$$\vec{\mathrm{X}} = \vec{\mathrm{X}}_-(\vec{r}, t)\,\mathrm{e}^{-i\omega t} + \vec{\mathrm{X}}_+(\vec{r}, t)\,\mathrm{e}^{i\omega t}. \tag{15}$$

On substituting Eqs. (14) and (15) into Eq. (13) and separately setting the coefficients of the exponential factors equal to zero, we obtain,

$$\left(i\hbar\frac{\partial}{\partial t} - V - \hbar\omega\right)\Psi_+ + (i\hbar c\vec{\nabla} + e\vec{A})\cdot\vec{\mathrm{X}}_+ = 0, \tag{16}$$

$$\left(i\hbar\frac{\partial}{\partial t} - V + \hbar\omega\right)\Psi_- + (i\hbar c\vec{\nabla} + e\vec{A})\cdot\vec{\mathrm{X}}_- = 0. \tag{17}$$

Dirac's equation ,

$$\left(i\hbar\frac{\partial}{\partial t} - V - mc^2\right)\psi + c\vec{\sigma}\cdot(i\hbar\vec{\nabla} + e\vec{A})\xi = 0 \tag{18}$$

$$\left(i\hbar\frac{\partial}{\partial t} - V + mc^2)\xi + c\vec{\sigma}\cdot(i\hbar\vec{\nabla} + e\vec{A}\right)\psi = 0, \tag{19}$$

follows immediately on setting $\hbar\omega = mc^2$, $\vec{\mathrm{X}}_+ = \vec{\sigma}\vec{\Psi}_-$, $\vec{\mathrm{X}}_- = \vec{\sigma}\vec{\Psi}_+$, $\Psi_+ = \psi$, $\Psi_- = \xi$. The reader may verify that Eqs. (18), (19) and (11) are identical on carrying out the matrix operations in Eq. (11) using

$$\psi_{\mathrm{D}} = \begin{pmatrix}\psi\\ \xi\end{pmatrix}$$

where ξ and ψ are known in the literature as the large and small components of the Dirac solution. Notice that the electron's spin can be interpreted as the polarization of the vector component of its posited 4-potential. The derivation suggests that the distinction between the material and electromagnetic properties of the electron, which I examined in a previous paper [20], may be an artificial one due to conceptual and practical limitations of theory but manifestly present in observations such as the Lamb shift or the electron's anomalous magnetic moment. Notice that no further proof of the Lorentz invariance of the wave equation itself is required since Eqs. (18) and (19) have been inferred directly from a scalar product of 4-vectors.

The standard 4-component solutions are separated into products of radial and angular solutions, thus,

$$\psi_{\mathrm{D}}(r,\theta,\phi) = \begin{pmatrix}\psi(r,\theta,\phi)\\ \xi\,(r,\theta,\phi)\end{pmatrix} = \begin{pmatrix} g_\kappa(r)\chi_{\kappa\mu}(\theta,\phi)\\ if_\kappa(r)\chi_{-\kappa\mu}(\theta,\phi)\end{pmatrix}, \tag{20}$$

where the relative phases are chosen so that the radial functions are real. The two-component spinors, $\chi_{\kappa\mu}(\theta,\phi)$, have equal and opposite quantum numbers forκ and$-\kappa$ due to the properties of the operator,

$$\vec{\sigma}\cdot\vec{\nabla} = \sigma\cdot\hat{r}\left(\frac{\partial}{\partial r} - \frac{1}{r}\vec{\sigma}\cdot\vec{\ell}\right),$$

where $\vec{\sigma}\cdot\vec{\ell}\chi_{\kappa\mu}(\theta,\phi) = -(\kappa+1)\chi_{\kappa\mu}(\theta,\phi)$ and $\vec{\sigma}\cdot\hat{r}\chi_{\kappa\mu}(\theta,\phi) = -\chi_{-\kappa\mu}(\theta,\phi)$. The spinors are eigen functions of the operators j^2, j_z, ℓ^2, and s^2, where$\vec{j}$ is the total angular momentum operator, $\vec{j} = \vec{\ell} + \vec{s}$, and therefore of $\vec{\sigma}\cdot\vec{\ell} = 2\vec{\ell}\cdot\vec{s} = j^2 - \ell^2 - s^2$ where the eigen values of j^2, ℓ^2, and s^2 are j (j+1), $\ell(\ell+1)$, and $\frac{1}{2}(\frac{1}{2}+1)$, respectively, such that the κ quantum number has values κ=-1, -2… for $j = \ell + \frac{1}{2}$ and κ =1, 2… for $j = |\ell - \frac{1}{2}|$.

The key point here is that the spinors are eigen functions of the angular-momentum operators listed above but not of $\vec{\sigma}\cdot\vec{\nabla}$ due to the diagonal nature of this operator in the z-coordinate , the axis along which the permanent magnetic moment due to the electron's spin is measured. Let us look at this operator in z, ρ, ϕ coordinates,

$$\vec{\sigma}\cdot\vec{\nabla} = \begin{pmatrix} \frac{\partial}{\partial z} & \mathrm{e}^{-i\phi}(\frac{\partial}{\partial\rho} - \frac{i}{\rho}\frac{\partial}{\partial\phi})\\ \mathrm{e}^{i\phi}(\frac{\partial}{\partial\rho} + \frac{i}{\rho}\frac{\partial}{\partial\phi}) & -\frac{\partial}{\partial z}\end{pmatrix} \tag{21}$$

The diagonal nature of $\vec{\sigma}\cdot\vec{\nabla}$ along z suggests that it may be possible for bound states having the Landau form to exist. Of course the standard set of states in r, θ, ϕ coordinates known in the literature must also be recoverable in z, ρ, ϕ coordinates using *the same set of physical boundary conditions* as used in the standard set of states. One would expect however for Landau-type states to have a different set of boundary conditions from the standard solutions in analogy to the different boundary conditions used in Section 2 for the small-size, low-energy states of the Klein–Gordon equation.

4. Dirac-equation States with Landau Symmetry

Dirac's equation should be cast in exact second-order form [21] for ease of analysis by elimination of Eq. (19) in favor of Eq. (18),

$$\left[(E-V)^2 - m^2c^4 + \hbar^2c^2\left(\nabla^2 + \frac{(\vec{\sigma}\cdot\vec{\nabla}V)(\vec{\sigma}\cdot\vec{\nabla})}{E-V+mc^2}\right)\right]\psi = 0, \tag{22}$$

where we return once again to the time-independent equation. First, let us analyze the equation in r, θ, ϕ coordinates where, using Eq. (9),

$$(\vec{\sigma}\cdot\vec{\nabla}V)(\vec{\sigma}\cdot\vec{\nabla}) = \vec{\nabla}V\cdot\nabla + i\vec{\sigma}\cdot(\vec{\nabla}V\times\vec{\nabla}) = V'\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\vec{\sigma}\cdot\vec{\ell}}{r}\right), \tag{23}$$

where the V' denotes the radial derivative of V. Specializing to $\ell = 0$ ($\kappa = -1$) states the last term on the right-hand side of Eq. (23) gives zero contribution. Neglecting $E+mc^2$compared to -V in Eq. (22) the small-r equation is

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{3}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\beta^2}{r^2}\right)g = 0, \tag{24}$$

which may be compared with Eq. (2) and whose indicial equation is $s(s-1)+3s+\beta^2 = 0$ with solutions

$$s = -1 \pm \sqrt{1-\beta^2} \simeq -\frac{1}{2}\beta^2, -2 + \frac{1}{2}\beta^2$$

for upper and lower signs, respectively. Notice that the lower-sign solution, in contrast to that of the Klein–Gordon equation, is unambiguously irregular at the origin and must be rejected as unphysical. The $V'(\mathrm{d}/\mathrm{d}r)$ term in Eq. (23) increases the kinetic energy near $r = 0$ beyond the physical bounds encompassed by a normalizable wave function. In contrast the Klein–Gordon wave function is infinite at $r = 0$, but it is still normalizable and therefore has a chance of being physically realizable in a realistic point-source situation.

Compared to the Schrödinger and Klein–Gordon equations it may seem fishy to the reader to have a V' or radial-force contribution to the kinetic energy, but that is the nature of the spin–orbit interaction. It may be possible, however, for a wave function to exist whose probability distribution is not radially concentrated at a single point in space but is cylindrically distributed along z in a cigar shape, which is the shape of the electronic distribution for an atom in a strong magnetic field [12]. Returning to Eq. (22) the numerator of the spin–orbit term in z, ρ, ϕ coordinates is

$$(\vec{\sigma}\cdot\vec{\nabla}V)(\vec{\sigma}\cdot\vec{\nabla}) = \begin{pmatrix} V_z\frac{\partial}{\partial z} + V_\rho\left(\frac{\partial}{\partial\rho} + \frac{i}{\rho}\frac{\partial}{\partial\phi}\right) & \mathrm{e}^{-i\phi}\left[V_z\left(\frac{\partial}{\partial\rho} - \frac{i}{\rho}\frac{\partial}{\partial\phi}\right) - V_\rho\frac{\partial}{\partial z}\right] \\ -\mathrm{e}^{i\phi}\left[V_z\left(\frac{\partial}{\partial\rho} + \frac{i}{\rho}\frac{\partial}{\partial\phi}\right) - V_\rho\frac{\partial}{\partial z}\right] & V_z\frac{\partial}{\partial z} + V_\rho\left(\frac{\partial}{\partial\rho} - \frac{i}{\rho}\frac{\partial}{\partial\phi}\right) \end{pmatrix} \tag{25}$$

where the subscripts on V denote first-order derivatives with respect to z or ρ. This operator can be represented by expanding the wave function in a basis set, $\{\psi_\nu(z, \rho, \phi)\}$, whose members comprise products of magnetic sub-states,

$$\psi_\nu(z, \rho, \phi) = \psi_{\alpha,m_z}(z, \rho)\,\mathrm{e}^{im_z\phi}\alpha + \psi_{\beta,m_z+1}(z, \rho)\,\mathrm{e}^{i(m_z+1)\phi}\beta, \tag{26}$$

where α, β are spin-up, spin-down spin states and $\nu = 1/2$ for $m_z = 0$, $\nu = 3/2$ for $m_z = 1$ and so on. We will examine only the azimuthally symmetric component, $\psi_{\alpha,0}$, here.

Again proceeding heuristically I propose a variational trial solution having the form $\psi_{\alpha,0} = N(\rho^s \pm az)^{-1}\mathrm{e}^{-w\rho^2}$ in the positive, negative domain of z for the upper, lower sign, where $a > 0$ to insure that the wave function is bounded in the z direction and $s < 1$ to insure that integrals with integrands which are singular as ρ^{-2s+1}, which occur in the expectation value of the potential, do not diverge logarithmically. Notice that these Landau-type states are bound in an elongated sense along the z direction and are bound radially in the standard exponentially decaying manner. Hence, these are bound states which are stabilized by spreading out the kinetic energy in an elongated, cigar shape rather than concentrating it at a single point, as for the Dirac solution in r, θ, ϕ coordinates.

The a parameter is determined from a generalized indicial equation to be given shortly. The s parameter is written $s = 1 - b$ where $b \ll 1$ and, along with w, is determined by minimizing the energy. The trial wave function is obviously square integrable,

$$N^2 \int_0^\infty \mathrm{d}\rho\, \rho\, \mathrm{e}^{-2w\rho^2} 2 \int_0^\infty \mathrm{d}z\, (\rho^s + az)^{-2} = 1, \tag{27}$$

where the z integral, along with all of the z integrals except one, are evaluated analytically. In Eq. (27) the z integral is equal to$\rho^{-s}a^{-1}$. The Gaussian form, $\mathrm{e}^{-w\rho^2}$, is used rather than the Slater form, $\mathrm{e}^{-w\rho}$, because the latter boosts the kinetic energy to an extent that binding is impossible.

The small z, smallρ equation for $\psi_{\alpha 0}$ from Eq. (22) is

$$\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2 + z^2}\left(z\frac{\partial}{\partial z} + \rho\frac{\partial}{\partial \rho}\right) + \beta^2 \frac{1}{\rho^2 + z^2}\right) g = 0, \tag{28}$$

where again $E + mc^2$ is neglected compared to $-V$. Substituting $g = (\rho^s \pm az)^{-1}$ into Eq. (28), carrying out the operations, multiplying the result by g, and finally integrating over z, an indicial equation is given by

$$\rho^{-3}\left\{\frac{1}{6a} + \frac{2a}{3} + (\beta^2 - 1)a(1 + a^2)^{-1}\left[1 + (1 + a^2)^{-1}\left[\ln a^2 + a^{-1}(1 - a^2)(\tan^{-1} a + \tan^{-1} a^{-1})\right]\right]\right\} = 0, \tag{29}$$

where without loss of accuracy b has been set equal to zero, which choice makes the factorization of ρ^{-3}possible, as shown in Eq. (29). Two values of a are found to satisfy Eq. (29), $a \simeq 0.19$ and $a \simeq 0.565$, for which values the energy is plotted versus w in Fig. 2.

The energy is found from Eq. (22),

$$E^2 + 2EZe^2\langle\psi|o_1|\psi\rangle - m^2c^4 + \hbar^2c^2(\langle\psi|(o_2 + o_3 + o_4 + o_5)|\psi\rangle) = 0, \tag{30}$$

with roots,

$$E = -Ze^2\langle\psi|o_1|\psi\rangle \pm \sqrt{(Ze^2\langle\psi|o_1|\psi\rangle)^2 + m^2c^4 - \hbar^2c^2(\langle\psi|o_2 + o_3 + o_4 + o_5|\psi\rangle)}, \tag{31}$$

where the potential-energy expectation value is

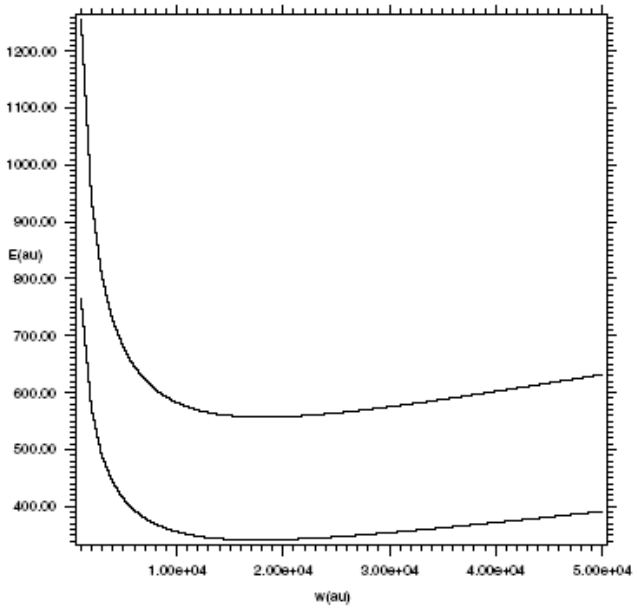

Figure 2. Energy versus variational parameter, w, for two Landau-type states of the Dirac equation for $b = \alpha^2$. Upper: $a = 0.19$. Lower: $a = 0.565$. The binding energy is $mc^2 - E$.

$$\begin{aligned}\langle\psi|o_1|\psi\rangle &= 2N^2 \int_0^\infty \mathrm{d}\rho\, \rho\, \mathrm{e}^{-2w\rho^2} \int_0^\infty \mathrm{d}z\, (\rho^s + az)^{-2}(\rho^2 + z^2)^{-1/2} \\ &\simeq 2N^2 c_p \int_0^\infty \mathrm{d}\rho\, \rho^{2b-1}\mathrm{e}^{-2w\rho^2} \\ &= \frac{N^2 c_p \rho^{2b}\mathrm{e}^{-w\rho^2}}{b}\bigg|_0^\infty + \frac{4N^2 c_p w}{b} \int_0^\infty \mathrm{d}\rho\, \rho^{2b+1}\mathrm{e}^{-2w\rho^2} \end{aligned} \tag{32}$$

$$c_\mathrm{p} = \frac{a-1}{1+a^2} + \frac{1}{(1+a^2)^{3/2}} \ln \frac{a\left[(1+a^2)^{1/2} + a\right]}{(1+a^2)^{1/2} - 1}. \tag{33}$$

The kinetic-energy expectation values are

$$\langle\psi|o_2|\psi\rangle = 2N^2 \int_0^\infty \mathrm{d}\rho\, \rho \left[(2w\rho)^2 - 2w\right] \mathrm{e}^{-2w\rho^2} \int_0^\infty \mathrm{d}z\, (\rho^s + az)^{-2} = 2N^2 a^{-1} \int_0^\infty \mathrm{d}\rho\, \rho^b \left[(2w\rho)^2 - 2w\right] \mathrm{e}^{-2w\rho^2} \tag{34}$$

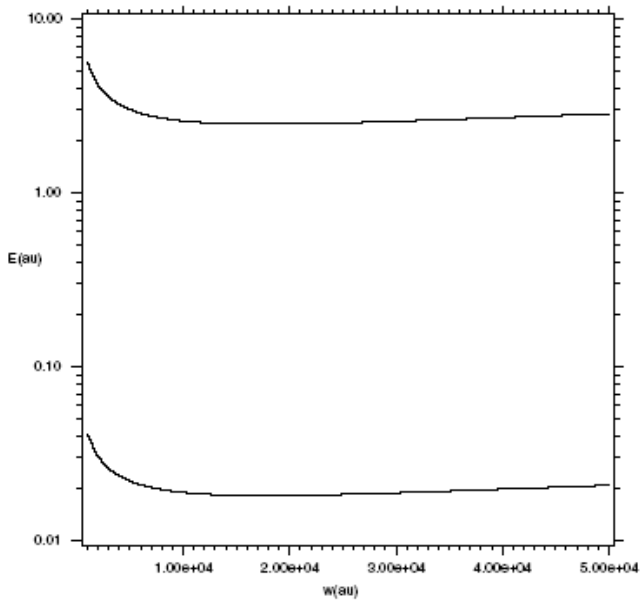

Figure 3. Energy versus variational parameter, w, for two Landau-type states of the Dirac equation for $a = 0.565$. Upper: $b = \alpha^3$. Lower: $b = \alpha^4$. The binding energy is $mc^2 - E$.

$$\langle\psi|o_3|\psi\rangle = 4N^2 w \int_0^\infty \mathrm{d}\rho\, \rho^2 \mathrm{e}^{-2w\rho^2} \int_0^\infty \mathrm{d}z\, (\rho^s + az)^{-3} = 2N^2 a^{-1} w \int_0^\infty \mathrm{d}\rho\, \rho^{2b} \mathrm{e}^{-2w\rho^2}, \tag{35}$$

$$\langle\psi|o_4|\psi\rangle = -4N^2 w \int_0^\infty \mathrm{d}\rho\, \rho\, \mathrm{e}^{-2w\rho^2} \int_0^\infty \mathrm{d}z\, (\rho^s + az)^{-2} = -2N^2 a^{-1} w \int_0^\infty \mathrm{d}\rho\, \rho^{b} \mathrm{e}^{-2w\rho^2}, \tag{36}$$

$$\langle\psi|o_5|\psi\rangle = -4N^2 wZe^2 \int_0^\infty \mathrm{d}\rho\, \rho^3 \mathrm{e}^{-2w\rho^2} \int_0^\infty \mathrm{d}z \frac{1}{(\rho^s + az)^2(\rho^2 + z^2)(Ze^2 + mc^2\sqrt{\rho^2 + z^2})}. \tag{37}$$

The integral over z in Eq. (36) is evaluated numerically. All integrals over ρ are evaluated numerically. Notice that $\langle\psi|o_3|\psi\rangle \simeq -\langle\psi|o_4|\psi\rangle$, such that these contributions to the kinetic energy nearly cancel. As in the case of the Klein–Gordon equation the binding energy is of order mc^2. Our solution of the Dirac equation requires two variational parameters, w and b, and Figs. 2 and 3 show that E has a minimum versus w for different values of b, while the energy approaches zero – binding energy of mc^2 – as b approaches zero. This behavior merely reflects the logarithmic divergence of the potential-energy expectation value for $b = 0$. But the rule for evaluating the integral in Eq. (32) is the take the limit as$\rho \to 0$ for finite b. For example using L'Hopital's rule for the opposite limit, $b \to 0$ for finite ρ,

$$\frac{\rho^{2b}}{b} = \frac{\mathrm{e}^{2b\ln\rho}}{b} \to 2\ln\rho.$$

This means that any physical problem for which the present theory is suitable must truly involve a point-Coulomb source such as an electron or positron. In the case of experiments whose interpretation appears to require the existence of quantum states whose energies are lower than those of the known states in the literature, to which I have given the generic name "hydrino states," this work is intended to establish that no general rule exists by which such states can be said to be incompatible with quantum mechanics. In the original hydrino-state work [1–7] no distinction was made between a point-nucleus and a finite-nucleus model of the hydrogen atom, although, as we now understand from the present analysis, this distinction is of critical importance.

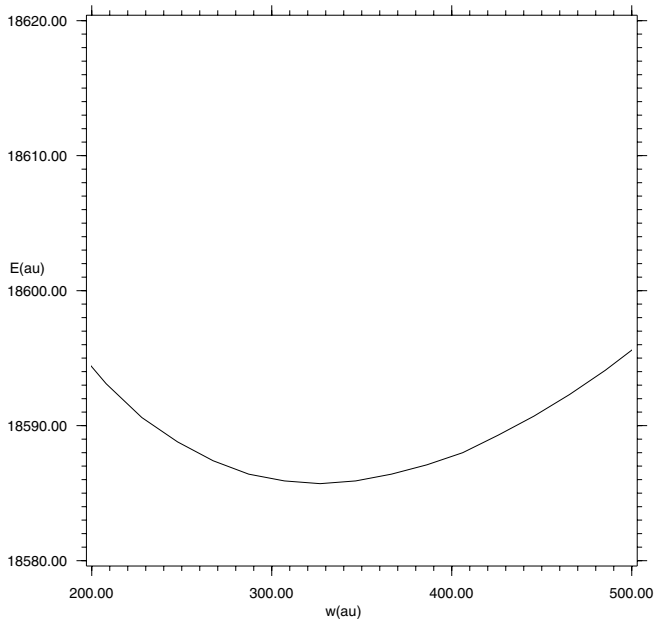

Figure 4. Energy versus variational parameter, w, for the negative-s state of the Klein–Gordon equation using a finite-proton potential model. The binding energy is $mc^2 - E$.

5. Finite-nucleus Potential Model

The interpretation of experimental data requires the use of finite-nucleus potential models in which the nuclear charge-distribution from the origin to the surface of the nucleus is represented. A well-studied model [22] is used, in which

$$V = \frac{-Ze^2}{2r_0}\left[3 - (r/r_0)^2\right] \quad \text{for} \quad r < r_0 \quad \text{and} \quad V = \frac{-Ze^2}{r} \quad \text{for} \quad r > r_0 \,,$$

where r_0 is the nuclear radius; for a proton $r_0 = 1.044 \times 10^{-13}$cm or 1.974×10^{-5}Å. No bound states were found for Dirac's equation using this model. The trial function $\psi_{\alpha,0} = N(\rho^s \pm az)^{-p}\mathrm{e}^{-w\rho^2}$ was used, where $\frac{1}{2} < p \leq 1$. For $p = 1$, which was used in the point-nucleus model in Section 4, the indicial equation appropriate for a finite proton did not have a solution for any value of a. A solution of the indicial equation for $p = 0.8$ was found for $a = 0.65$, but this solution had no bound states. With reference to Eq. (31) the reason appeared to be the lack of cancellation of $\langle\psi|o_3|\psi\rangle$ and $\langle\psi|o_4|\psi\rangle$ for $p = 0.8$, giving a kinetic energy which exceeds the potential energy.

The Klein–Gordon equation was revisited using the finite-proton model, and the trial function $\psi = Nr^{-1}\mathrm{e}^{-wr}$ (first trial function) was used, for which the kinetic energy was negative and therefore unphysical. Then the trial function $\psi = Nr^{-1}\mathrm{e}^{-wr^2}$ (second trial function) was used, for which a bound state was found whose energy is plotted versus w in Fig. 4. The binding energy at the minimum is in the 5 keV range. Notice that the Schrödinger irregular form r^{-1} may be used since the potential is now finite at $r = 0$. With reference to Eq. (4), the kinetic energy is negative for the first trial function due to the exact cancellation of the $2\frac{\mathrm{d}}{\mathrm{d}r}r^{-1}\frac{\mathrm{d}}{\mathrm{d}r}\mathrm{e}^{-wr}$ and $\frac{2}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\mathrm{e}^{-wr}$ contributions to the kinetic energy. This cancellation also occurs using the second trial function, but the term $r^{-1}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\mathrm{e}^{-wr^2}$ sufficiently boosts the kinetic energy to avert total cancellation of the m^2c^4 term. The latter situation leads to the dominance of the matrix element of V^2, which is negative. Recall that V^2 is removed in the indicial equation in the point-nucleus model. The magnitude of the energy in Fig. 4 is large owing to the smallness of matrix elements of V in the finite-proton model.

6. Conclusions

In summary, the hydrino states exist for both the Klein–Gordon and the Dirac equations, but they are sensitive to using a point-nucleus versus a finite-nucleus model, critically so in Dirac's equation and less so in the Klein–Gordon equation. Since the Klein–Gordon equation cannot sensibly be proposed as an EOM for the electron, the negative result for Dirac's equation suggests that evidence of excess energy production in experiments likely cannot be explained as due to the existence of an unusual quantum state.

What might be considered for further investigation using Dirac-equation anomalous quantum states is the nature of the positron-electron state as a real two-fermion state rather than as a two-fermion-state concept in order to interpret Dirac's equation. The latter concept encompasses the standard interpretation of Dirac's equation, not as an equation for a single particle, but as an equation for an infinite number of particles, providing a practical calculational tool possibly at the cost of providing an idealization of physical reality. The Bethe–Salpeter equation [23] for two fermions was a move in the direction of an actual many-fermion theory, but its mathematical complexity appears to put it beyond the reach of realistic practical calculations. A recent quantum-dynamical approach [24] proposed by the present author might be worth pursuing since the two-fermion Coulomb interaction can be represented, as discussed there, without loss of Lorentz invariance.

Appendix A. Reviewer Comments

It is proposed that Ritchie's paper be published. However, since there remain issues, it seems appropriate to develop a brief set of reviewer's comments to be made available for the benefit of those reading the paper. I will summarize some

of these issues in what follows.

Appendix A.1. Lorentz invariance

Ritchie has included some discussion about using a Lorentz-invariant model to address the associated problems. One approach to this would be to include the nucleus in the calculation in the framework of a relativistic two-body problem, making use of a Bethe–Salpeter or related type of model. The resulting model would be explicitly Lorentz-invariant. Relativistic field theory was constructed to allow for such problems to be analyzed systematically. An electron in a fixed central field potential would generally not be regarded as a Lorentz- invariant model.

Appendix A.2. Coulomb point source model

Ritchie argues that the problem of an electron and positron would provide an example of a system involving two point source charges, since the nuclear charge is spread out on the Fermi scale. I note that vacuum polarization effects lead to a spreading out of the effective charge of electrons and positrons, also on the Fermi scale, so that in the end there is no qualitative difference between the problems.

Appendix A.3. Occupation of negative energy states

Ritchie reminds us of the (somewhat anachronistic) view that negative energy Dirac states are occupied, which was put forth (about 80 years ago) by analogy with electrons and holes in semiconductors. In more modern times, the positive energy solutions are adopted to describe electrons, and the negative energy states are used to describe positrons; as in the construction of QED. Within such a picture there cannot be real electron occupation of a negative energy Dirac state, so the issue of transitions to such states does not arise.

Appendix A.4. Use of the variational principle

Ritchie has made use of a variational calculation under conditions where there is not a lowest energy state. Although under some conditions reasonable results can be obtained, in general one does not have confidence in the result of such a computation. It would have been better to make use of a modified variational method, perhaps based the minimization of $I\,[\psi] = \langle\psi|(H - E_{\rm off})^2|\psi\rangle/\langle\psi|\psi\rangle$ where $E_{\mathbf{off}}$ is selected to be close to the target state energy.

Appendix A.5. Use of the mixed symmetry states

In the case of the Dirac equation, Ritchie uses a variational wave function with mixed symmetry. In a central potential one would expect states with pure (or unmixed) symmetry to be Eigen functions. It would have been better to focus on a single channel with fixed symmetry.

Appendix A.6. Localized $S_{1/2}$ state

I provided Ritchie with an analytic result for a localized $S_{1/2}$ state in the case of the Dirac equation. If one starts with $E\psi = \beta mc^2 + \alpha \cdot c_{\rm p} + V(r)\psi$ and works with states of the form

$$\psi = \begin{pmatrix} \frac{P(r)}{r}\chi_{k,m}(\theta,\phi) \\ \\ i\,\frac{Q(r)}{r}\chi_{-k,m}(\theta,\phi) \end{pmatrix}$$

then the Dirac equation for large and small components in the case of a point Coulomb model reduces to

$$E_P = mc^2 P - \hbar c \left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{k}{r}\right) Q - \frac{e^2}{r} P,$$

$$E_Q = -mc^2 Q + \hbar c \left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{k}{r}\right) P - \frac{e^2}{r} Q$$

It is possible to develop exact solutions based on the ansatz

$$P(r) = r^s \mathrm{e}^{-\beta r} \quad \text{and} \quad Q(r) = A r^s \mathrm{e}^{-\beta r}.$$

After plugging in, we get two $S1/2$ state solutions. In the case of the ground state $1S_{1/2}$ state, we end up with

$$s = \sqrt{1-\alpha^2}$$

$$P(r) = r^{\sqrt{1-\alpha^2}} \mathrm{e}^{-r/a_0} \quad \text{and} \quad Q(r) = A r^{\sqrt{1-\alpha^2}} \mathrm{e}^{-r/a_0},$$

$$E = mc^2 + \frac{\hbar c}{\alpha a_0}(s+k) = mc^2 + 2I_H \frac{s+k}{\alpha^2} \approx mc^2 - I_H$$

We also get a localized state with

$$s = -\sqrt{1-\alpha^2}$$

$$P(r) = r^{-\sqrt{1-\alpha^2}} \mathrm{e}^{-r/a_0} \quad \text{and} \quad Q(r) = A r^{-\sqrt{1-\alpha^2}} \mathrm{e}^{-r/a_0},$$

$$E = mc^2 + \frac{\hbar c}{\alpha a_0}(s+k) = mc^2 + 2I_H \frac{s+k}{\alpha^2} \approx -mc^2$$

We have found a localized $\mathrm{S}_{1/2}$state in the case of a point Coulomb potential. We recognize this as a negative energy state.

Appendix A.7. Acceptance of QED

Quantum electrodynamics (QED) was constructed as a relativistic quantum theory capable of addressing problems involving electrons, positrons, and photons. Extensions of QED that treat protons as Dirac particles have been used for high-precision calculations for atomic hydrogen. As a theory QED by now has achieved many striking successes, and is sometimes called the most accurate physical theory.

Due to the close connection between the positive energy Dirac states and the spectrum of states predicted in QED, we would not expect electrons in localized (negative energy) states around a proton. We would also not expect an electron to be able to make a transition to a negative energy Dirac orbital since these are not part of the state space available to electrons in the theory.

There are a number of proposals for localized electronic states under discussion in the literature. In light of the comments here, almost all such models would then be immediately at variance with QED. The electron orbitals might be derived from a Klein–Gordon equation, or from some other model, which at the outset would not be consistent with QED. Alternatively, the orbital might come from a localized solution of the Dirac equation; which if made up of positive energy states must have a large positive kinetic energy; and if made up of negative energy states is excluded in QED.

It would seem that a model which seeks to account for cold fusion effects based on electronic transitions to negative energy Dirac states, to states predicted by the Klein–Gordon equation, or to other states inconsistent with QED, ultimately implies that either QED is deficient in some way (that one is proposing to improve QED itself in a fundamental way). I will point out that if one thinks that the road is hard dealing with anomalies connected with the Fleischmann–Pons experiment, the road associated with arguing that QED as a theory is wrong in some fundamental way will be a thousand times hard

Author's Response: General Considerations

I wish to emphasize that the approach in this paper is not based on the methodology of standard quantum electrodynamics (QED). A considerable literature exists on different electro-dynamical theories proposed for the calculation of the radiative properties of matter, for which, in order better to orient the reader to the subject matter of this paper, I give a brief review as follows. The quantization of the classical electromagnetic field was carried out by Dirac in 1927 [25]. A review of the quantized radiation field (QRF), as it is called, and its use in the calculation of radiative spontaneous emission and the Lamb shift is given by Louisell [26]. The QRF may be criticized in the sense that its distribution of frequencies is unrelated to the electron's own distribution of frequencies and is therefore unbounded such that its use in the radiation-matter interaction Hamiltonian for the electron leads to an energy shift – Lamb shift – which diverges linearly in the photon frequency, ω. As explained in [26] and elsewhere the linear divergence is interpreted as a permanent radiant property of a free electron such that, when it is included or "added back" to the calculation for a bound electron which is "bare" or undressed by the radiation field in the original calculation, the linear divergence is exactly canceled. This procedure is known as mass renormalization. Although a logarithmic divergence in the photon frequency remains, use of a suitable cut off leads to results, which agree quantitatively with experiment [27,28]. Notice that the linearly divergent mass of a free electron appears to be irremovable.

In order to gain a more satisfactory physical picture of the radiant aspect of the electron, pioneers have presented formulations – the neoclassical theory of Edwin Jaynes and coworkers [28] and self-field quantum electrodynamics of Asim Barut and coworkers [29] – in which the electromagnetic vector potential is calculated from the electron's current. These theories were problematic either in a quantitative sense in Jaynes' case or in the sense of possible flaws in the use of Schrödinger theory to calculate the electron's current in Barut's case [30,31].

The quantization condition for the photon and electron [25,26] requires that an initial higher-energy state of the electron has zero photons and that a final lower-energy state of the electron has one photon. The radiative emission rate converges because it vanishes by destructive interference of the out-of-phase electron wave functions of the initial and final states unless $\hbar\omega = \Delta E_{fi}$, where ΔE_{fi} is the energy gap between the two states and $\hbar\omega$ is the photon energy. On the other hand emission of a photon from the ground state means that the photon must be re-absorbed by the same state leading to a closed photon loop in which the electron energy shift diverges as ω. Dirac's relativistic-electron equation [32] leads to further complications in interpretation since a set of negative-energy states lies below the nominal ground state such that radiative spontaneous emission from the ground state to a negative-energy state lying below it would occur, which is unobserved in nature. Dirac's hole interpretation that the negative-energy states are filled with electrons in which an absent electron or hole represents a positron avoids the unphysical prediction since a positive-energy electron is forbidden by Pauli's exclusion principle from falling into a negative-energy state. But Dirac's hole theory also rules out the existence of Zitterbewegung, which arises from the interference between positive-energy and negative-energy states in observables in which an electron simultaneously occupies a superposition of positive-energy and negative-energy states. The recent observation of Zitterbewegung in a simulated electron experiment using a trapped-ion [33] suggests that hole theory, for all its success in positron physics, should be reexamined from the point of view of its possible reconciliation with Zitterbewegung. What is the ground state? A body of theory exists known as 4-space Dirac theory [34,35] in which the positive-energy spectrum of states is identical to that of standard Dirac theory but in which the wave function comprises contributions from both electrons and positrons, which one may surmise is just a bound-state form of Zitterbewegung, although not identified as such likely owing to the fact that the original prediction

of Zitterbewegung [36] was for a free electron. Following Barut and coworkers [29,37] and others, it is necessary here to pursue a first-quantization approach in order to understand phenomena usually treated within second quantization.

Recent work [34] suggests that the negative-energy states do not lie empty below the ground state but rather actively participate with it to form a two-component ground-state configuration. If the negative-energy states do not lie empty below the positive-energy states, then the quantization rules for radiative spontaneous emission do not physically apply. In [34, 35] the positive-energy spectrum is identical to that of standard Dirac theory, but the wave function exhibits Zitterbewegung (or comprises contributions from both electrons and positrons in the post-hole language of [35]). But the original motivation and experimental confirmation of Dirac theory was the spectroscopic observation of atomic fine structure. *Thus standard Dirac theory and 4-space Dirac theory* [34,35] *are therefore both confirmed by spectroscopic experiments, such that the confirmation of wave-function Zitterbewegung predicted by 4-space theory requires experiments designed to probe the wave function and not the energy spectrum*. In short Dirac hole theory is incompatible with the experimental observation of Zitterbewegung, which exists if indeed the negative-energy states do not lie empty below the positive-energy states such that radiative spontaneous emission from the *nominally* positive-energy ground state cannot exist and therefore does not need to be blocked by the artifice of filling up the negative-energy levels with electrons.

It seems clear from the above discussion that, while the QRF is physically correct for radiative spontaneous emission, it has unphysical consequences for the radiative shift of energy levels, which is corrected in practical applications by using the physical argument of mass renormalization. Indeed in his original paper [25] Dirac *limits* the use of the QRF to the emission and absorption of radiation and the derivation of the Einstein A and B coefficients. But one can use the renormalization concept that an electron permanently has radiant properties, which are therefore always present, such that the concept of a bare electron loses meaning. In standard QED this concept takes the form of continuously emitted and reabsorbed photons by the quantum state of a free electron, whose mathematical implementation, as stated above, leads to the divergent shift linear in ω for a free electron – the divergence which, when included in the bound-electron calculation, cancels the divergent shift linear in ω for the bound electron. One may postulate that this concept can be realized by finding a first-quantized Lorentz-invariant relativisitic equation of motion which accounts for the radiant properties of the electron in the same way that Dirac's equation accounts for the material properties of the electron. A small literature using the concept of a photon EOM already exists [26], but its applications appear to be confined to experiments in which the radiation-matter interaction is unimportant.

The shift of an atomic energy level relative to its position as predicted by radiation-free quantum mechanics and as observed experimentally, suggests that radiation is a permanent component of atomic structure. But existing theory, which comprises a quantum theory of matter, a quantum theory of radiation, and a quantum theory of radiation-matter interaction, perversely forces one to discard this conclusion in favor of a quantum-field-theoretic picture in which photons are created and destroyed relative to a quantum vacuum state [25,26]. Second, quantization extends this picture to particle fields, but in this paper we will be concerned only with first quantization for electron states. We can propose as well a first quantization picture for the quantum states of radiation. This is a departure from the approach outlined in [38], which used quantum field theory to propose a photon EOM. Our motivation is clear. The quantization of the radiation field by Dirac [25] to describe the emission and absorption of radiation fails to describe radiative energy shifts in absence of the use of renormalization theory to remove infinite contributions. The concept of radiation as a permanent part of the quantum states of the electron is actually introduced in renormalization theory, as discussed above. But again the inexorable field-theoretic logic of the continuous emission and absorption of virtual photons by the *same quantum state in a closed photon loop* leads to a radiative correction to the electron's mass which diverges as ω [26]. The missing concept whose mathematical implementation would avoid this failure uses the logic that the quantum states of matter *exist simultaneously and permanently with the quantum states of radiation* such that the artificial boundary-value setup of virtual-photon emission and absorption is avoided. Since the quantum states of matter are given by Dirac's equation, we require a supplemental wave equation to give the quantum states of radiation associated with the electron.

While Dirac's equation accounts for the material properties of the electron, the supplemental radiation wave equation may be considered to account for the radiant properties of the electron, as observed experimentally in the Lamb shift and the electron's anomalous magnetic moment.

It is easy to propose a radiant equation of motion (REOM) for the electron once it is recognized that the electron's material equation of motion (MEOM), which is Dirac's equation, can be inferred from the scalar product of the electron's 4-momentum and a material 4-potential posited for the electron. This understanding of Dirac's equation suggests that a REOM can be inferred from the photon's 4-momentum and an electromagnetic 4-potential posited for the electron.Recalling that the scalar product of 4-vectors is always Lorentz invariant [39], Dirac's equation can be derived by further elucidating the close relationship between Dirac's equation and the spinorial form of Maxwell's equation, which has been studied continuously since 1928 [40–43]. Dirac's equation can be inferred from the scalar product of the electron's 4-momentum and a material 4-potential, $(\Psi, \vec{X})$, posited for the electron as follows,

$$\left(i\hbar\frac{\partial}{c\partial t} - \frac{V}{c}, i\hbar\vec{\nabla} + \frac{e}{c}\vec{A}\right)(\Psi, \vec{X}) = \left(i\hbar\frac{\partial}{c\partial t} - \frac{V}{c}\right)\Psi + \left(i\hbar\vec{\nabla} + \frac{e}{c}\vec{A}\right)\vec{X}) = 0. \tag{A.1}$$

Using a carrier-wave expansions for $(\Psi, \vec{\mathrm{X}})$ in order to isolate a dominant frequency component of the 4-potential we obtain,

$$\Psi = \Psi_{-}(\vec{r}, t)\,\mathrm{e}^{-i\omega t} + \Psi_{+}(\vec{r}, t)\,\mathrm{e}^{i\omega t} \tag{A.2}$$

$$\vec{\mathrm{X}} = \vec{\mathrm{X}}_{-}(\vec{r}, t)\,e^{-i\omega t} + \vec{\mathrm{X}}_{+}(\vec{r}, t)\,\mathrm{e}^{i\omega t}. \tag{A.3}$$

On substituting Eqs. (A.2) and (A.3) into Eq. (A.1) and separately setting the coefficients of the exponential factors equal to zero, I obtain,

$$\left(i\hbar\frac{\partial}{\partial t} - V - \hbar\omega\right)\Psi + c\vec{\sigma}\cdot(i\hbar c\vec{\nabla} + e\vec{A})\xi = 0, \tag{A.4}$$

$$\left(i\hbar\frac{\partial}{\partial t} - V + \hbar\omega\right)\xi + c\vec{\sigma}\cdot(i\hbar c\vec{\nabla} + e\vec{A})\psi = 0. \tag{A.5}$$

Dirac's equation,

$$\left(i\hbar\frac{\partial}{\partial t} - V - mc^2)\psi + c\vec{\sigma}\cdot(i\hbar\vec{\nabla} + e\vec{A}\right)\xi = 0 \tag{A.6}$$

$$\left(i\hbar\frac{\partial}{\partial t} - V + mc^2)\xi + c\vec{\sigma}\cdot(i\hbar\vec{\nabla} + e\vec{A}\right)\psi = 0 \tag{A.7}$$

follows immediately on setting $\hbar\omega = mc^2,\ \vec{X}_{+} = \vec{\sigma}\,\Psi_{-},\ \vec{X}_{-} = \sigma\,\Psi_{+},\ \Psi_{+} = \psi,\ \Psi_{-} = \xi$. The reader may verify that Eqs. (A.6) and (A.7) are indeed Dirac's equation in coupled first-order form where ψ and ξ are known in the literature as the large and small components of the Dirac solution, respectively. Notice that the electron's spin can be interpreted as the polarization of the vector component of its posited material 4-potential. Notice that no further proof of the Lorentz invariance of the wave equation itself is required since Eqs. (A.6) and (A.7) have been inferred directly from a

scalar product of 4-vectors, which is always a Lorentz invariant [39]. As an example Dirac's equation for a hydrogen-like ion [44] is manifestly Lorentz invariant, but a fully relativistic Lorentz-invariant theory for two fermions is given by the Bethe–Salpeter equation [45]. As a further complication Coulomb's law for the interelectronic interaction is incompatible with Lorentz invariance such that it is represented field-theoretically by the exchange of virtual photons. If indeed future experiments show that Zitterbewegung is a real physical effect arising from the simultaneous occupancy of both positive-energy and negative-energy states by an electron, then the Bethe–Salpeter equation describing a positron and electron should be appealed to for a proper description of annihilation and pair creation.

While Eqs. (A.6) and (A.7) account for atomic fine structure and the anomalous Zeeman effect, whose spectroscopic observation was the motivation for Dirac's equation and its experimental confirmation, radiant properties of the electron also exist which are observed as a quantum electro-dynamical shift of atomic energy levels and the electron's anomalous magnetic moment. It is assumed that an electromagnetic 4-potential exists for the electron such that a REOM can be inferred from the Lorentz invariant found from the scalar product of the photon's 4-momentum and the electron's posited electromagnetic 4-potential thus,

$$\left(\frac{\hbar}{c}\frac{\partial}{\partial t}, \hbar\vec{\nabla} - \frac{e\hbar}{mc^2}\vec{E}, \vec{H}\right)\cdot(\Phi_v, \vec{A}_v) = \frac{\hbar}{c}\frac{\partial}{\partial t}\Phi_e + \left(\hbar\vec{\nabla} - \frac{e\hbar}{mc^2}\vec{E}, \vec{H}\right)\cdot\vec{A}_v = 0, \tag{A.8}$$

for either electric or magnetic fields $\vec{E}, \vec{H}$. The photon four-momentum is found from $\hbar$times a 4-gradient,

$$\left(\frac{\partial}{c\partial t}, \vec{\nabla} - \frac{e}{mc^2}\vec{E}, \vec{H}\right),$$

whose scalar product with the electromagnetic 4-current,

$$c\left(u + \int_0^t \mathrm{d}t'\vec{j}\cdot\vec{E}\right), \vec{S},$$

where

$$u = \frac{1}{8\pi}(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B})$$

is the electromagnetic energy density and

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{H}$$

is the electromagnetic 3-current, gives the Lorentz-invariant electromagnetic continuity equation,

$$\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} + \vec{j}\cdot\vec{E} = 0. \tag{A.9}$$

This is simply the electromagnetic analog of writing the Lorentz-invariant material continuity equation,

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0, \tag{A.10}$$

as the scalar product of the known 4-gradient,

$$\left(\frac{\partial}{c\partial t}, \vec{\nabla}\right),$$

and the known material 4-current, $(c\rho, \vec{j})$. Notice that in the radiant-electron theory developed above the known 4-gradient is simply renormalized by the replacement

$$\vec{\nabla} \to \vec{\nabla} - \frac{e}{mc^2}\vec{E}, \vec{H},$$

which gives a Lorentz-invariant electromagnetic continuity equation since the scalar product of $\vec{E}$ or $\vec{H}$ with the electromagnetic 3-current, $\vec{S}$, vanishes. It is remarkable that a photon 4-momentum seems not to have been previously proposed in the literature.

As with the electron the photon scalar and vector potentials can be written in the form of carrier-wave expansions,

$$\Phi_\nu = \Phi_{\nu-}\mathrm{e}^{-i\omega_\nu t} + \Phi_{\nu+}\mathrm{e}^{i\omega_\nu t} \tag{A.11}$$

$$\vec{A}_\nu = \vec{A}_{\nu-}\mathrm{e}^{-i\omega_\nu t} + \vec{A}_{\nu+}\mathrm{e}^{i\omega_\nu t}, \tag{A.12}$$

from which on substituting Eqs. (A.11) and (A.12) into Eq. (A.8) and separately setting the coefficients of the exponential factors equal to zero, we obtain,

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + i\frac{\omega_\nu}{c}\right)\Phi_{\nu+} + \left(\vec{\nabla} - \frac{e}{mc^2}\vec{E}, \vec{H}\right)\cdot\vec{A}_{\nu+} = 0, \tag{A.13}$$

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - i\frac{\omega_\nu}{c}\right)\Phi_{\nu-} + \left(\vec{\nabla} - \frac{e}{mc^2}\vec{E}, \vec{H}\right)\cdot\vec{A}_{\nu-} = 0. \tag{A.14}$$

On setting

$$\Phi_{\nu+} = \xi_{E,H}, \quad \vec{A}_{\nu+} = \vec{\sigma}\zeta_{E,H}, \quad \Phi_{\nu-} = \zeta_{E,H}, \quad \vec{A}_{\nu-} = \vec{\sigma}\xi_{E,H}$$

we obtain the

Dirac form for the REOM,

$$\frac{\partial\xi_{E,H}}{c\partial t} + i\frac{\omega_\nu}{c}\xi_{E,H} + \vec{\sigma}\cdot\left(\vec{\nabla} - \frac{e}{mc^2}\vec{E}, \vec{H}\right)\zeta_{E,H} = 0 \tag{A.15}$$

$$\frac{\partial\zeta_{E,H}}{c\partial t} - i\frac{\omega_\nu}{c}\zeta_{E,H} + \vec{\sigma}\cdot\left(\vec{\nabla} - \frac{e}{mc^2}\vec{E}, \vec{H}\right)\xi_{E,H} = 0. \tag{A.16}$$

Writing

$$\xi_{E,H} = \mathrm{e}^{-i\omega t}\psi_{E,H} \quad \text{and} \quad \zeta_{E,H} = \mathrm{e}^{-i\omega t}\chi_{E,H}$$

in Eqs. (A.15) and (A.16) we derive stationary equations for $\psi_{E,H}$ and $\chi_{E,H}$; then we eliminate the equation for $\chi_{E,H}$ in favor of a second-order equation for $\psi_{E,H}$, obtaining equations for the electric and magnetic photon wave functions which have the Helmholtz form,

$$\left\{\nabla^2 + \frac{\omega^2 - \omega_\nu^2}{c^2} - \frac{e}{mc^2}\left[\vec{\nabla}\cdot\vec{E} + 2\vec{E}\cdot\vec{\nabla} + i\vec{\sigma}\cdot(\vec{\nabla}\times\vec{E}) - \frac{e}{mc^2}E^2\right]\right\}\psi_E = 0, \tag{A.17}$$

$$\left\{\nabla^2 + \frac{\omega^2 - \omega_\nu^2}{c^2} - \frac{e}{mc^2}\left[\vec{\nabla}\cdot\vec{H} + 2\vec{H}\cdot\vec{\nabla} + i\vec{\sigma}\cdot(\vec{\nabla}\times\vec{H}) - \frac{e}{mc^2}H^2\right]\right\}\psi_H = 0, \tag{A.18}$$

where we have used the identity,

$$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = \vec{A}\cdot\vec{B} + i\vec{\sigma}\cdot(\vec{A}\times\vec{B}).$$

Equation (A.18), for $\hbar\omega_\nu = 0$, was used in physical applications to calculate a *divergence-free* Lamb shift [46] and electron's anomalous magnetic moment [47].

Notice that all four of Maxwell's equations appear in Eqs. (A.17) and (18) as radiation-matter interaction terms and that the electromagnetic fields themselves and not the electromagnetic potentials occur such there is no question of a gauge dependence of matter-light interactions in the electron's REOM. The success of the use of Eqs. (A.17) and (18) to calculate divergence-free radiative properties of matter [46,47] suggests that the concept of radiation as a permanent part of the structure of matter is a valid one. Recall that this is identically the concept of mass renormalization used in standard QED used to remove infinite contributions to the electron's energy arising from unphysical logic that *first-quantized* states of matter exist which are totally free of radiation. As I have shown here it is possible to present a theory in which the electron does not exist in a bare or radiation-free state and whose material and radiant properties are described by a pair of relativistic, Lorentz-invariant first-quantized material and radiant EOM's respectively.

Acknowledgements

The author is grateful to T. Scott Carman for supporting this work. This work was performed under the auspices of the Lawrence Livermore National Security, LLC, (LLNS) under Contract No. DE-AC52-07NA27344.

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