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Research Article

Maxwell's Equations and Occam's Razor

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Abstract

In this paper a straightforward application of Occam's razor principle to Maxwell's equation shows that only one entity, the electromagnetic four-potential, is at the origin of a plurality of concepts and entities in physics. The application of the so called "Lorenz gauge" in Maxwell's equations *denies the status of real physical entity* to a scalar field that has a gradient in space-time with clear physical meaning: the four-current density field. The mathematical formalism of space-time Clifford algebra is introduced and then used to encode Maxwell's equations starting only from the electromagnetic four-potential. This approach suggests a particular *Zitterbewegung* (ZBW) model for charged elementary particles.

Keywords: Coulomb gauge, Clifford algebra, Electric charge, Electron structure, Elementary particles, Maxwell's equations, Lorenz gauge, Occam's razor, Space–time algebra, Vector potential, *Zitterbewegung*

Nomenclature (see p. 101)

1. Introduction

Science is the creation and validation of models of abstract concepts and experimental data. For this reason it is important to examine the rules used to evaluate the quality of a model. Occam's razor principle emphasizes the simplicity and conciseness of the model: among different models that fit experimental data, the simplest one must

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Nomenclature

Symbol	Name	SI units	Natural units (NU)
$\boldsymbol{A}_{\square}$	Electromagnetic four-potential	$\mathrm{V\ s\ m^{-1}}$	eV
$\boldsymbol{r}_{\square}$	Four-position vector	m	$\mathrm{eV^{-1}}$
$\boldsymbol{G}$	Electromagnetic field	$\mathrm{V\ s\ m^{-2}}$	$\mathrm{eV^2}$
$\boldsymbol{F}$	Electromagnetic field bivector	$\mathrm{V\ s\ m^{-2}}$	$\mathrm{eV^2}$
$\boldsymbol{B}$	Flux density field	$\mathrm{V\ s\ m^{-2} = T}$	$\mathrm{eV^2}$
$\boldsymbol{E}$	Electric field	$\mathrm{V\ m^{-1}}$	$\mathrm{eV^2}$
S	Scalar field	$\mathrm{V\ s\ m^{-2}}$	$\mathrm{eV^2}$
$\boldsymbol{J}_{\square e}$	Four-current density	$\mathrm{A\ m^{-2}}$	$\mathrm{eV^3}$
$\boldsymbol{v}_{\square}$	Four-velocity vector	$\mathrm{m\ s^{-1}}$	1
$\boldsymbol{A}'$	Electromagnetic eight-potential	$\mathrm{V\ s\ m^{-1}}$	eV
P	Pseudoscalar field	$\mathrm{V\ s\ m^{-2}}$	$\mathrm{eV^2}$
$\boldsymbol{J}_{\square \mathrm{m}}$	Magnetic four-current density	$\mathrm{A\ s\ m^{-3}}$	$\mathrm{eV^3}$
ρ	Electric charge density	$\mathrm{A\ s\ m^{-3} = C\,m^{-3}}$	$\mathrm{eV^3}$
ρ_{m}	Magnetic charge density	$\mathrm{A\ m^{-2}}$	$\mathrm{eV^3}$
x, y, z	Space coordinates	m $^{(1)}$	$\mathrm{eV^{-1}}$
t	Time variable	s $^{(2)}$	$\mathrm{eV^{-1}}$
c	Light speed in vacuum	$2.99792458 \times 10^8\ \mathrm{m\ s^{-1}}$	1
μ_0	Permeability of vacuum	$4\pi \times 10^{-7}\ \mathrm{V\ s\ A^{-1}\,m^{-1}}$	4π
ϵ_0	Dielectric constant of vacuum	$8.854187817 \times 10^{-12}\ \mathrm{A\ s\ (V\ m)^{-1}}$	$1/4\pi$
$\boldsymbol{P}_{\square}$	Electromag. four-momentum	$\mathrm{kg\ m\ s^{-1}}$	eV
$\mathfrak{S}$	Generalized Poynting vector	$\mathrm{W\ m^{-2}}$	$\mathrm{eV^4}$
w	Specific energy	$\mathrm{J\ m^{-3}}$	$\mathrm{eV^4}$

$^{(1)}$ $1.9732705 \times 10^{-7}\ \mathrm{m} \approx 1\ \mathrm{eV^{-1}}$;
$^{(2)}$ $6.5821220 \times 10^{-16}\ \mathrm{s} \approx 1\ \mathrm{eV^{-1}}$.

be preferred, i.e. the model that does not introduce concepts or entities that are not strictly necessary. The following sentences in Latin briefly illustrate this principle:

Pluralitas non est ponenda sine necessitate.
Frustra fit per plura quod potest fieri per pauciora.
Entia non sunt multiplicanda praeter necessitatem.

[1], which can be translated respectively as "plurality should not be posited without necessity", "it is futile to do with more things that which can be done with fewer" and "entities must not be multiplied beyond necessity".

According to this principle, the quality of a model can be measured by means of two fundamental parameters:

(1) Good agreement of a model's predictions with experimental data and/or with other expected results.
(2) The simplicity of a model, a value that is inversely related to the amount of information, concepts, entities, exceptions, postulates, parameters and variables used by the model itself.

These rules are universal ones and can be applied in many contexts [2]. From this point of view, the intuitive and simple framework of Clifford algebra is a natural choice.

In this paper, we introduce and use the space–time Clifford algebra, showing that only one fundamental physical entity is sufficient to describe the origin of electromagnetic fields and charges, i.e. the electromagnetic four-potential. The vector potential should not be viewed only as a mathematical tool but as a real physical entity, as suggested by the Aharonov–Bohm effect, a quantum mechanical phenomenon in which a charged particle is affected by the vector potential in regions in which the electromagnetic fields are null [3]. Actually, many papers deal with the application of geometric algebra to Maxwell's equation (see [4–9] and many others), but few of them deal with the concept of scalar field. Among the most interesting works we can find a paper by Bettini [5], two papers written by van Vlaenderen [10,11] and two papers of Hively [12,13].

In this paper we propose a reinterpretation of Maxwell's equations which does not use any gauge: the unique constraint is that the electromagnetic four-potential must be represented by a *harmonic* function, as proposed by Bettini [5]. This fact gives rise to an electromagnetic field composed not only of the classical electric and magnetic flux density fields, but also by a scalar field. The scalar field will be here investigated and it will be shown that its existence produces many interesting implications and consequences on the essence of electrical charges and currents. A brief and simple but essential introduction on the main fundamental properties of Clifford algebra is given preliminarily in this paper in order to encourage a particular interpretation of Maxwell's equations at the picometric scale.

This paper is composed of the following parts: Section 2 is a short introduction to Clifford algebra and its fundamental properties; Section 3 illustrates how Maxwell's equations can be derived from a four dimensional vector potential without using the Lorenz gauge; Section 4 deals with the main properties of the electromagnetic field, the derivation of Maxwell's equations from the Lagrangian density, the Lorentz force, the generalized Poynting vector, the symmetrical Maxwell's equations and, finally, in Section 5 some essential points are summarized.

2. The Language of Scientific Knowledge

Scientific knowledge is expressed mathematically, but the importance of the optimal choice of the appropriate mathematical language is often underestimated [4–6,14]. The geometric algebra (Clifford algebra) formalism, according to Occam's razor principle, is by far the best choice for modern physics. Clifford algebra provides a simple and unifying mathematical language for coding geometric entities and operations [8,9,15]. It integrates different mathematical concepts highlighting geometrical meanings that are often hidden in the ordinary algebra.

A particular Clifford $Cl_{p,q}$ algebra is defined in a space with $n = p + q$ dimensions with an orthonormal base of n unitary vectors. The first p vectors of this base have positive squares, whereas the remaining ones have negative squares, as shown by the following equations:

$$\gamma_i^2 = 1 \text{ with } 1 \leq i \leq p, \tag{1}$$

$$\gamma_i^2 = -1 \text{ with } p + 1 \leq i \leq p + q, \tag{2}$$

$$\gamma_i \gamma_j = -\gamma_j \gamma_i \text{ with } i \neq j, \tag{3}$$

where γ_i are the unitary orthogonal vectors.

The geometrical product $\gamma_i \gamma_j$ represents a "segment" of a unitary "area" of undefined shape in the plane identified by unitary vectors γ_i and γ_j. The product $\gamma_1 \gamma_2 ... \gamma_n$ represents a unitary, *n*-dimensional volume segment identified by

Table 1. Blades of space–time algebra ($Cl_{3,1}$).

Blade	Bit mask	Grade	hex.
1	0000	0 (scalar)	0
γ_x	0001	1 (vector)	1
γ_y	0010	1 (vector)	2
$\gamma_x\gamma_y = \gamma_{xy}$	0011	2 (bivector)	3
γ_z	0100	1 (vector)	4
$\gamma_x\gamma_z = \gamma_{xz}$	0101	2 (bivector)	5
$\gamma_y\gamma_z = \gamma_{yz}$	0110	2 (bivector)	6
$\gamma_x\gamma_y\gamma_z = \gamma_{xyz} = I_\triangle$	0111	3 (pseudovector)	7
γ_t	1000	1 (vector)	8
$\gamma_x\gamma_t = \gamma_{xt}$	1001	2 (bivector)	9
$\gamma_y\gamma_t = \gamma_{yt}$	1010	2 (bivector)	A
$\gamma_x\gamma_y\gamma_t = \gamma_{xyt}$	1011	3 (pseudovector)	B
$\gamma_z\gamma_t = \gamma_{zt}$	1100	2 (bivector)	C
$\gamma_x\gamma_z\gamma_t = \gamma_{xzt}$	1101	3 (pseudovector)	D
$\gamma_y\gamma_z\gamma_t = \gamma_{yzt}$	1110	3 (pseudovector)	E
$\gamma_x\gamma_y\gamma_z\gamma_t = \gamma_{xyzt} = I$	1111	4 (pseudoscalar)	F

the unitary vectors $\gamma_1,\ \gamma_2,\ \ldots,\ \gamma_n$. In the *n*-dimensional space no more than 2^n elementary distinct "components" exist. Each of these entities corresponds to a particular subset of the orthonormal base vectors. The "grade" of these entities (called blades) is equal to the number of base vectors which are present within the subset. The blade of grade zero (empty set) is the dimensionless scalar unit. The number of blades of *k*th-grade in a *n*-dimensional space is equal to the binomial coefficient

$$N_k = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}. \tag{4}$$

Using Clifford algebra there are two possible choices for the metric of space–time coordinates, namely $Cl_{1,3}$ and $Cl_{3,1}$. For $Cl_{1,3}$ (signature "$+\ -\ -\ -$") we have

$$\gamma_t^2 = -\gamma_x^2 = -\gamma_y^2 = -\gamma_z^2 = 1,$$

whereas for $Cl_{3,1}$ (signature "$+\ +\ +\ -$")

$$\gamma_x^2 = \gamma_y^2 = \gamma_z^2 = -\gamma_t^2 = 1.$$

In $Cl_{3,1}$ algebra, which is used in this work, the spatial coordinates can be viewed as the familiar Cartesian coordinates of the Euclidean space. The 2^4 components of the $Cl_{3,1}$ space–time Clifford algebra are listed in Table 1.

Each blade is associated to a real number, the blade value. The expression $a\gamma_i\gamma_j$, in which a is a real scalar, represents an area a of an undefined shape in the plane identified by vectors γ_i and γ_j.

Summing can be carried out only between coefficients of identical blades. If used for different blades the sum must be reinterpreted as a composition, in analogy with the concept of sum between real and imaginary parts of a complex number.

A multi-vector is a generic composition of one or more blades. Within an *n*-dimensional space a multi-vector is composed by no more than 2^n blades. The geometrical product between two blades gives a blade which is obtained from the application of the *bitwise exclusive* OR operation between the bit mask of blades to be multiplied. The value of the resultant blade is equal to the product of the values of the operands multiplied by a sign which is a function of the two blades. For $Cl_{3,1}$ algebra the sign can be easily determined by means of Table 2, identifying the blades to be multiplied through their hexadecimal "label" taken from Table 1. In general, this product is not commutative. The sign table is obtained by the direct application of Eqs. (1)–(3).

There are other types of products in Clifford algebra, two of them are the wedge (symbol $\wedge$) and the scalar product (symbol $\cdot$). The result is computed following the same rules of geometric product, but is zero in some cases:

(1) the wedge product is always zero if the intersection between the set of base vectors of the first operand blade and the set of base vectors of the second operand blade is not empty;
(2) the scalar product is always zero if the blade of the first operand is different from that of the second operand.

The geometric product of two vectors in Clifford algebra can be decomposed in a scalar product and a wedge product according to the relation

$$\boldsymbol{uv} = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \wedge \boldsymbol{v}. \tag{5}$$

It is important to note that the space–time algebra of the four γ_i vectors is isomorphic to the algebra of Majorana matrices. The Majorana matrices are the Dirac gamma matrices times the imaginary unit.

Example 2.1. Some examples of the application of products are here reported referring to $Cl_{3,1}$:

$$\gamma_x\gamma_y\,\gamma_x\gamma_y = -\gamma_x\gamma_x\gamma_y\gamma_y = -1,$$

$$\gamma_x\gamma_y \cdot \gamma_z\gamma_t = 0,$$

Table 2. Signs of the geometrical product in the space–time algebra $Cl_{3,1}$.
$I = \gamma_{xyzt} = \gamma_x\gamma_y\gamma_z\gamma_t$, $I_\triangle = \gamma_{xyz} = \gamma_x\gamma_y\gamma_z$.

Grade	0	1	1	2	1	2	2	3	1	2	2	3	2	3	3	4
hex.	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
Label	1	γ_x	γ_y	γ_{xy}	γ_z	γ_{xz}	γ_{yz}	$I_\triangle$	γ_t	γ_{xt}	γ_{yt}	γ_{xyt}	γ_{zt}	γ_{xzt}	γ_{yzt}	I
1	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
γ_x	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
γ_y	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−
γ_{xy}	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−
γ_z	+	−	−	+	+	−	−	+	+	−	−	+	+	−	−	+
γ_{xz}	+	−	−	+	+	−	−	+	+	−	−	+	+	−	−	+
γ_{yz}	+	+	−	−	+	+	−	−	+	+	−	−	+	+	−	−
$I_\triangle$	+	+	−	−	+	+	−	−	+	+	−	−	+	+	−	−
γ_t	+	−	−	+	−	+	+	−	−	+	+	−	+	−	−	+
γ_{xt}	+	−	−	+	−	+	+	−	−	+	+	−	+	−	−	+
γ_{yt}	+	+	−	−	−	−	+	+	−	−	+	+	+	+	−	−
γ_{xyt}	+	+	−	−	−	−	+	+	−	−	+	+	+	+	−	−
γ_{zt}	+	+	+	+	−	−	−	−	−	−	−	−	+	+	+	+
γ_{xzt}	+	+	+	+	−	−	−	−	−	−	−	−	+	+	+	+
γ_{yzt}	+	−	+	−	−	+	−	+	−	+	−	+	+	−	+	−
I	+	−	+	−	−	+	−	+	−	+	−	+	+	−	+	−

$$a\gamma_x\gamma_y \wedge b\gamma_z = ab\gamma_x\gamma_y\gamma_z,$$

$$\gamma_x\gamma_y \wedge \gamma_y = 0,$$

$$\gamma_x\gamma_y \wedge \gamma_z\gamma_t = \gamma_x\gamma_y\gamma_z\gamma_t,$$

$$\gamma_x \cdot \gamma_y = 0,$$

$$(\gamma_x + \gamma_t)^2 = (\gamma_x + \gamma_t)(\gamma_x + \gamma_t) = \gamma_x^2 + \gamma_x\gamma_t + \gamma_t\gamma_x + \gamma_t^2 = 1 + \gamma_x\gamma_t - \gamma_x\gamma_t - 1 = 0$$

(example of light-like vector, the square is 0),

$$(\gamma_x + \gamma_y)^2 = (\gamma_x + \gamma_y)(\gamma_x + \gamma_y) = \gamma_x^2 + \gamma_x\gamma_y + \gamma_y\gamma_x + \gamma_y^2 = 1 + \gamma_x\gamma_y - \gamma_x\gamma_y + 1 = 2$$

(example of space-like vector, the square is > 0),

$$(a\gamma_t)^2 = -a^2$$

(example of time-like vector, the square is < 0),

$$(a\gamma_x + b\gamma_y)^2 = (a\gamma_x + b\gamma_y)(a\gamma_x + b\gamma_y) = a^2\gamma_x^2 + ab\gamma_x\gamma_y + ba\gamma_y\gamma_x + b^2\gamma_y^2 = a^2 + ab\gamma_x\gamma_y - ab\gamma_x\gamma_y + b^2 = a^2 + b^2$$

(always a space-like vector),

$$(a\gamma_x + b\gamma_t)^2 = (a\gamma_x + b\gamma_t)(a\gamma_x + b\gamma_t) = a^2\gamma_x^2 + ab\gamma_x\gamma_t + ba\gamma_t\gamma_x + b^2\gamma_t^2 = a^2 + ab\gamma_x\gamma_t - ab\gamma_x\gamma_t - b^2 = a^2 - b^2$$

(light-like if $a = b$, time-like if $a < b$, space-like if $a > b$).

In these examples a and b are generic real scalars.

2.1. Reflection and rotation of vectors

In order to perform a mirror reflection of a vector with respect to a plane, the following formula holds in Clifford algebra:

$$\boldsymbol{a}' = -\boldsymbol{mam}, \tag{6}$$

where $\boldsymbol{m}$ is the unitary vector orthogonal to surface α, as shown in Fig. 1. As a matter of fact, if

$$\boldsymbol{a} = \boldsymbol{a}_\perp + \boldsymbol{a}_\parallel,$$

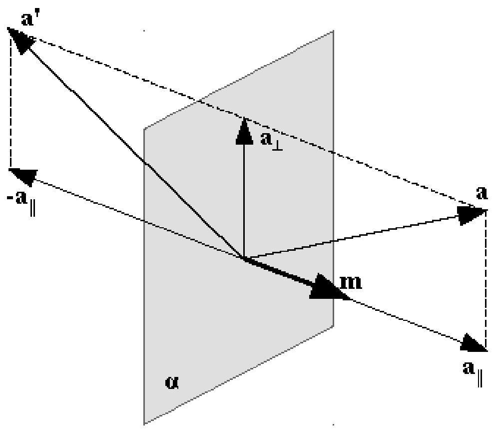

Figure 1. Reflection of a vector with respect to plane α.

where $\boldsymbol{a}_\perp$ and $\boldsymbol{a}_\parallel$ are the orthogonal and the parallel components of vector $\boldsymbol{a}$ respectively, then

$$\begin{aligned}\boldsymbol{a}' &= -\boldsymbol{m}\left(\boldsymbol{a}_\perp + \boldsymbol{a}_\parallel\right)\boldsymbol{m} = -\boldsymbol{m}\boldsymbol{a}_\perp\boldsymbol{m} - \boldsymbol{m}\boldsymbol{a}_\parallel\boldsymbol{m}\\ &= \boldsymbol{a}_\perp\boldsymbol{m}^2 - \boldsymbol{a}_\parallel\boldsymbol{m}^2 = \boldsymbol{a}_\perp - \boldsymbol{a}_\parallel.\end{aligned}$$

This operation is justified by the fact that the product between parallel vectors commutes, i.e.

$$\boldsymbol{m}\boldsymbol{a}_\parallel = \boldsymbol{a}_\parallel\boldsymbol{m},$$

whereas the product between orthogonal vectors anti-commutes, i.e.

$$\boldsymbol{m}\boldsymbol{a}_\perp = -\boldsymbol{a}_\perp\boldsymbol{m}.$$

If vector $\boldsymbol{a}'$ in now reflected again with respect unitary vector $\boldsymbol{n}$ rotated with respect to $\boldsymbol{m}$ by an angle $\theta/2$ we obtain

$$\boldsymbol{a}'' = \boldsymbol{n}\boldsymbol{a}'\boldsymbol{n} = \boldsymbol{n}\boldsymbol{m}\boldsymbol{a}\boldsymbol{m}\boldsymbol{n}. \tag{7}$$

Vector $\boldsymbol{a}''$ is rotated, with respect to vector $\boldsymbol{a}$, by an angle θ on the common plane of the two-vector $\boldsymbol{m}$ and $\boldsymbol{n}$ as shown in Fig. 2.

The rotation of vector $\boldsymbol{a}$ can be described also by the following formula

$$\boldsymbol{a}'' = \boldsymbol{R}\boldsymbol{a}\tilde{\boldsymbol{R}} = \mathrm{e}^{-\boldsymbol{b}\frac{\theta}{2}}\boldsymbol{a}\mathrm{e}^{\boldsymbol{b}\frac{\theta}{2}}, \tag{8}$$

where $\boldsymbol{R} = \boldsymbol{n}\boldsymbol{m}$, $\tilde{\boldsymbol{R}} = \boldsymbol{m}\boldsymbol{n}$ and the bivector $\boldsymbol{b} = \boldsymbol{m}\wedge\boldsymbol{n}$ is a segment of the surface on which vector $\boldsymbol{a}$ is rotated.

The product of two vectors is called rotor. We remember that

$$\boldsymbol{R} = \boldsymbol{n}\boldsymbol{m} = \boldsymbol{m}\cdot\boldsymbol{n} - \boldsymbol{m}\wedge\boldsymbol{n}$$

and

$$\tilde{\boldsymbol{R}} = \boldsymbol{m}\boldsymbol{n} = \boldsymbol{m}\cdot\boldsymbol{n} + \boldsymbol{m}\wedge\boldsymbol{n}$$

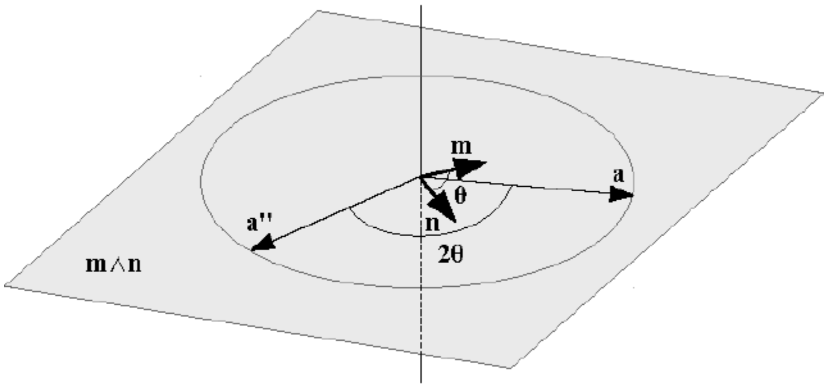

Figure 2. Rotation of vector $\boldsymbol{a}$ due to two subsequent reflections

It is important to note that these rules are independent from the signature, and for this reason they are also valid for four-vectors of the space–time algebra. In particular, rotors with pure spatial bivector parts (such as $\gamma_x\gamma_y\vartheta/2$) generate ordinary rotations, whereas rotors containing bivectors with the term γ_t (such as $\gamma_z\gamma_t\vartheta/2$) generate hyperbolic rotations. Rotors operations are a very powerful geometric tool and some hardware implementations have been attempted [16].

3. The Electromagnetic Field and the Wave Function

The behavior of electromagnetic waves was described in 1865 by James Clerk Maxwell in his work "Dynamical Theory of the Electromagnetic Field". Maxwell's equations are a system of partial differential equations, where different concepts are employed: electric field, flux density (or magnetic) field, charge density and current density [4–6].

In order to study the undulatory behavior of particles, the concept of wave function was introduced. Following the interpretation of Born, the square of this function represents the probability density to find a particle in a point of the space, just like the undulatory theory of light, whose intensity is given by the square of the electromagnetic wave amplitude. Now, following the principle of Occam's razor, which suggests carefulness in the introduction of new concepts, we consider two interesting possibilities:

(1) find a common origin of the conceptual entities used in Maxwell's equations;
(2) consider the wave function as a particular reformulation of concepts/entities already present in Maxwell's equations.

3.1. The electromagnetic potential

Maxwell's equations can be reinterpreted by means of a unique entity, namely, the vector potential with four components, as defined by the following equation:

$$\boldsymbol{A}_\square\left(\boldsymbol{r}_\square\right) = \gamma_x A_x\left(\boldsymbol{r}_\square\right) + \gamma_y A_y\left(\boldsymbol{r}_\square\right) + \gamma_z A_z\left(\boldsymbol{r}_\square\right) + \gamma_t A_t\left(\boldsymbol{r}_\square\right), \tag{9}$$

where each of the vector potential components A_x, A_y, A_z and A_t are functions of the space–time coordinates and $\boldsymbol{r}_\square\left(x,y,z,t\right) = \gamma_x x + \gamma_y y + \gamma_z z - \gamma_t ct = \boldsymbol{r}_\triangle - \gamma_t ct$ is the position vector in space–time. From now on in the four-potential and in other field quantities the variable $\boldsymbol{r}_\square$ will be omitted for simplicity. The four-potential has dimension in SI units equal to V s m^{-1}. Two basic assumptions are made:

(1) the vector potential field $\boldsymbol{A}_\square$ is represented by a *harmonic* function;
(2) the space is homogeneous, linear and isotropic.

Therefore, we assume a function that links a vector of four components to each point of the space–time as the unique source of Maxwell's equations entities.

We use the following definition of the operator $\boldsymbol{\partial}$ in space–time algebra

$$\boldsymbol{\partial} = \gamma_x \frac{\partial}{\partial x} + \gamma_y \frac{\partial}{\partial y} + \gamma_z \frac{\partial}{\partial z} + \gamma_t \frac{1}{c}\frac{\partial}{\partial t} = \nabla + \gamma_t \frac{1}{c}\frac{\partial}{\partial t}, \tag{10}$$

where

$$\nabla = \gamma_x \frac{\partial}{\partial x} + \gamma_y \frac{\partial}{\partial y} + \gamma_z \frac{\partial}{\partial z} \quad \text{and} \quad c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}.$$

If $\boldsymbol{A}_\square$ is the vector potential defined by (9) the following expression can be written:

$$\boldsymbol{\partial} \boldsymbol{A}_\square = \boldsymbol{\partial} \cdot \boldsymbol{A}_\square + \boldsymbol{\partial} \wedge \boldsymbol{A}_\square = S + \boldsymbol{F} = \boldsymbol{G}, \tag{11}$$

where

$$\boldsymbol{G}(x,y,z,t) = S + \gamma_x\gamma_t F_{xt} + \gamma_y\gamma_t F_{yt} + \gamma_z\gamma_t F_{zt} + \gamma_y\gamma_z F_{yz} + \gamma_x\gamma_z F_{xz} + \gamma_x\gamma_y F_{xy}. \tag{12}$$

Expanding (11), by considering the products as shown in Table 3 and by collecting all terms with the same blade, the following set of equations is found:

$$\boldsymbol{\partial} \cdot \boldsymbol{A}_\square = S = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} - \frac{1}{c}\frac{\partial A_t}{\partial t}, \tag{13}$$

$$\gamma_x\gamma_t F_{xt} = \gamma_x\gamma_t \frac{1}{c} E_x = \gamma_x\gamma_t \left(\frac{\partial A_t}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial t} \right), \tag{14}$$

$$\gamma_y\gamma_t F_{yt} = \gamma_y\gamma_t \frac{1}{c} E_y = \gamma_y\gamma_t \left(\frac{\partial A_t}{\partial y} - \frac{1}{c}\frac{\partial A_y}{\partial t} \right), \tag{15}$$

$$\gamma_z\gamma_t F_{zt} = \gamma_z\gamma_t \frac{1}{c} E_z = \gamma_z\gamma_t \left(\frac{\partial A_t}{\partial z} - \frac{1}{c}\frac{\partial A_z}{\partial t} \right), \tag{16}$$

$$\gamma_y\gamma_z F_{yz} = \gamma_y\gamma_z B_x = \gamma_y\gamma_z \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \right), \tag{17}$$

$$\gamma_x\gamma_z F_{xz} = -\gamma_x\gamma_z B_y = \gamma_x\gamma_z \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right), \tag{18}$$

Table 3. Products $\boldsymbol{\partial A}_\square$.

$\boldsymbol{\partial A}_\square$	$\gamma_x A_x$	$\gamma_y A_y$	$\gamma_z A_z$	$\gamma_t A_t$
$\gamma_x \frac{\partial}{\partial x}$	$\frac{\partial A_x}{\partial x}$	$\gamma_x\gamma_y \frac{\partial A_y}{\partial x}$	$\gamma_x\gamma_z \frac{\partial A_z}{\partial x}$	$\gamma_x\gamma_t \frac{\partial A_t}{\partial x}$
$\gamma_y \frac{\partial}{\partial y}$	$-\gamma_x\gamma_y \frac{\partial A_x}{\partial y}$	$\frac{\partial A_y}{\partial y}$	$\gamma_y\gamma_z \frac{\partial A_z}{\partial y}$	$\gamma_y\gamma_t \frac{\partial A_t}{\partial y}$
$\gamma_z \frac{\partial}{\partial z}$	$-\gamma_x\gamma_z \frac{\partial A_x}{\partial z}$	$-\gamma_y\gamma_z \frac{\partial A_y}{\partial z}$	$\frac{\partial A_z}{\partial z}$	$\gamma_z\gamma_t \frac{\partial A_t}{\partial z}$
$\gamma_t \frac{1}{c} \frac{\partial}{\partial t}$	$-\gamma_x\gamma_t \frac{1}{c} \frac{\partial A_x}{\partial t}$	$-\gamma_y\gamma_t \frac{1}{c} \frac{\partial A_y}{\partial t}$	$-\gamma_z\gamma_t \frac{1}{c} \frac{\partial A_z}{\partial t}$	$-\frac{1}{c} \frac{\partial A_t}{\partial t}$

$$\gamma_x\gamma_y F_{xy} = \gamma_x\gamma_y B_z = \gamma_x\gamma_y \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right), \tag{19}$$

where $S = S_1 + S_2 + S_3 + S_4$ is a scalar field, whose meaning will be clarified later. It is to be noted that equating (13) to zero, i.e. $S = 0$, gives an expression that takes the form of the "Lorenz gauge" condition if $A_t = -\varphi/c$, where φ is the scalar potential of the electric field [4,8,10,17].

Equation (13) can be rewritten as

$$S = \nabla \cdot \boldsymbol{A}_\triangle - \frac{1}{c}\frac{\partial A_t}{\partial t}, \tag{20}$$

where $\boldsymbol{A}_\triangle = \gamma_x A_x + \gamma_y A_y + \gamma_z A_z$ is the usual three-component vector potential.

Using the so-called "Lorenz gauge" the scalar field S is considered zero everywhere, *denying its status of a real physical entity* [5]. Same consideration can be done for the "Coulomb gauge" that assign zero value to each addendum S_i. We simply do not apply any "gauge", apart from defining $\boldsymbol{A}_\square$ as a harmonic function. According to our point of view, both Lorenz and Coulomb "gauges" should be considered just as boundary conditions and the scalar field S, although not directly observable, has a gradient in space–time with a clear physical meaning. Similar considerations are normally presented in electromagnetism to introduce the concept of vector potential, that is a not directly measurable field. The components of the geometric product $\boldsymbol{\partial A}_\square$ are shown in Table 3. An electromagnetic field $\boldsymbol{G}$ with seven components emerges, composed by one scalar and six bivectors.

Table 4 represents the relation between the fundamental electromagnetic entities and the space–time components of the vector potential $\boldsymbol{A}_\square$.

Table 4. Relation between electromagnetic entities and the vector potential $\boldsymbol{A}_\square$.

$\boldsymbol{\partial A}_\square$	$\gamma_x A_x$	$\gamma_y A_y$	$\gamma_z A_z$	$\gamma_t A_t$
$\gamma_x \frac{\partial}{\partial x}$	S_1	B_{z1}	$-B_{y1}$	$\frac{1}{c}E_{x1}$
$\gamma_y \frac{\partial}{\partial y}$	B_{z2}	S_2	B_{x1}	$\frac{1}{c}E_{y1}$
$\gamma_z \frac{\partial}{\partial z}$	$-B_{y2}$	B_{x2}	S_3	$\frac{1}{c}E_{z1}$
$\gamma_t \frac{1}{c} \frac{\partial}{\partial t}$	$\frac{1}{c}E_{x2}$	$\frac{1}{c}E_{y2}$	$\frac{1}{c}E_{z2}$	S_4

The set of equations from (14) to (19) can be rewritten also in the following way:

$$E_x = c\frac{\partial A_t}{\partial x} - \frac{\partial A_x}{\partial t} \tag{21}$$

$$E_y = c\frac{\partial A_t}{\partial y} - \frac{\partial A_y}{\partial t} \tag{22}$$

$$E_z = c\frac{\partial A_t}{\partial z} - \frac{\partial A_z}{\partial t} \tag{23}$$

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \tag{24}$$

$$B_y = -\frac{\partial A_z}{\partial x} + \frac{\partial A_x}{\partial z} \tag{25}$$

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}, \tag{26}$$

where

$$\boldsymbol{E} = \gamma_x E_x + \gamma_y E_y + \gamma_z E_z = c\nabla A_t - \frac{\partial \boldsymbol{A}_\triangle}{\partial t}, \tag{27}$$

$$\boldsymbol{B} = \gamma_x B_x + \gamma_y B_y + \gamma_z B_z = \nabla \times \boldsymbol{A}_\triangle. \tag{28}$$

The sum of all diagonal elements in Table 3 represents the scalar product

$$S = \boldsymbol{\partial} \cdot \boldsymbol{A}_\square, \tag{29}$$

whereas the sum of all extra-diagonal elements gives the six components of electromagnetic bivector $\boldsymbol{F}$

$$\boldsymbol{F} = \boldsymbol{\partial} \wedge \boldsymbol{A}_\square. \tag{30}$$

Referring to the function $\boldsymbol{G}$, it is possible to note that the "electromagnetic field" is characterized by seven values: three for the electric field, three for the flux density field and one for the scalar field S.

With reference to Table 4 the electromagnetic field $\boldsymbol{G}$ can also be expressed as

$$\begin{aligned}\boldsymbol{G}(x,y,z,t) &= S + \boldsymbol{F} = S + \gamma_x\gamma_t\frac{E_x}{c} + \gamma_y\gamma_t\frac{E_y}{c} + \gamma_z\gamma_t\frac{E_z}{c} + \gamma_y\gamma_z B_x - \gamma_x\gamma_z B_y + \gamma_x\gamma_y B_z \\ &= S + \frac{1}{c}\boldsymbol{E}\gamma_t + I\boldsymbol{B}\gamma_t = S + \frac{1}{c}(\boldsymbol{E} + Ic\boldsymbol{B})\gamma_t, \end{aligned} \tag{31}$$

where

$$I = \gamma_x \gamma_y \gamma_z \gamma_t \tag{32}$$

is the unitary pseudoscalar and

$$\boldsymbol{F} = \frac{1}{c}\boldsymbol{E}\gamma_t + I\boldsymbol{B}\gamma_t = \frac{1}{c}\left(\boldsymbol{E} + Ic\boldsymbol{B}\right)\gamma_t. \tag{33}$$

On the other hand, with reference to Table 3, the electromagnetic field $\boldsymbol{G}$ can be expressed in the following compact form

$$\boldsymbol{G}\left(x,y,z,t\right) = \nabla\cdot\boldsymbol{A}_{\vartriangle} - \frac{1}{c}\frac{\partial A_t}{\partial t} + \nabla A_t\gamma_t - \frac{1}{c}\frac{\partial \boldsymbol{A}_{\vartriangle}}{\partial t}\gamma_t + I\nabla\times\boldsymbol{A}_{\vartriangle}\gamma_t, \tag{34}$$

which again results in Eqs. (20), (27) and (28) by taking (31) into account.

3.2. Maxwell's equations

Now, by applying the operator $\boldsymbol{\partial}$ to the multivector $\boldsymbol{G}$ (11) and equating it to zero, a new expression is found, i.e.

$$\boldsymbol{\partial G} = \boldsymbol{\partial}^2\boldsymbol{A}_{\square} = 0, \tag{35}$$

whose components are shown in Table 5. The equation $\boldsymbol{\partial G} = 0$ *can be seen as an extension in four dimensions of the Cauchy-Riemann conditions for analytic functions of a complex (two dimensional) variable* [15,18]. In [18] Hestenes writes: *"Members of this audience will recognize* $\square\psi_0 = 0$ *as a generalization of the Cauchy–Riemann equations to space–time, so we can expect it to have a rich variety of solutions. The problem is to pick out those solutions with physical significance."*. In fact, if $\boldsymbol{A}_{\square}$ is harmonic then

$$\boldsymbol{\partial}^2\boldsymbol{A}_{\square} = \nabla^2\boldsymbol{A}_{\square} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}_{\square}}{\partial t^2} = 0, \tag{36}$$

which represents the wave equation of the four-potential and where

$$\boldsymbol{\partial}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}.$$

It should be noted that in our case, considering the scalar field $S \neq 0$ and $\boldsymbol{A}_{\square}$ harmonic, (36) is always homogeneous.

By collecting all common factors contained in Table 5 the following equations are derived:

$$\gamma_x\left(\frac{\partial S}{\partial x} - \frac{\partial B_z}{\partial y} + \frac{\partial B_y}{\partial z} + \frac{1}{c^2}\frac{\partial E_x}{\partial t}\right) = 0, \tag{37}$$

$$\gamma_y\left(\frac{\partial B_z}{\partial x} + \frac{\partial S}{\partial y} - \frac{\partial B_x}{\partial z} + \frac{1}{c^2}\frac{\partial E_y}{\partial t}\right) = 0, \tag{38}$$

$$\gamma_z\left(-\frac{\partial B_y}{\partial x} + \frac{\partial B_x}{\partial y} + \frac{\partial S}{\partial z} + \frac{1}{c^2}\frac{\partial E_z}{\partial t}\right) = 0, \tag{39}$$

Table 5. Products $\boldsymbol{\partial G} = \boldsymbol{\partial}\left(\boldsymbol{\partial A}_{\square}\right)$. $\gamma_{ij} = \gamma_i\gamma_j, \gamma_{ijk} = \gamma_i\gamma_j\gamma_k$.

$\boldsymbol{\partial}^2\boldsymbol{A}_{\square}$	S	$\gamma_{xt}\frac{1}{c}E_x$	$\gamma_{yt}\frac{1}{c}E_y$	$\gamma_{zt}\frac{1}{c}E_z$	$\gamma_{yz}B_x$	$-\gamma_{xz}B_y$	$\gamma_{xy}B_z$
$\gamma_x\frac{\partial}{\partial x}$	$\gamma_x\frac{\partial S}{\partial x}$	$\gamma_t\frac{1}{c}\frac{\partial E_x}{\partial x}$	$\gamma_{xyt}\frac{1}{c}\frac{\partial E_y}{\partial x}$	$\gamma_{xzt}\frac{1}{c}\frac{\partial E_z}{\partial x}$	$\gamma_{xyz}\frac{\partial B_x}{\partial x}$	$-\gamma_z\frac{\partial B_y}{\partial x}$	$\gamma_y\frac{\partial B_z}{\partial x}$
$\gamma_y\frac{\partial}{\partial y}$	$\gamma_y\frac{\partial S}{\partial y}$	$-\gamma_{xyt}\frac{1}{c}\frac{\partial E_x}{\partial y}$	$\gamma_t\frac{1}{c}\frac{\partial E_y}{\partial y}$	$\gamma_{yzt}\frac{1}{c}\frac{\partial E_z}{\partial y}$	$\gamma_z\frac{\partial B_x}{\partial y}$	$\gamma_{xyz}\frac{\partial B_y}{\partial x}$	$-\gamma_x\frac{\partial B_z}{\partial y}$
$\gamma_z\frac{\partial}{\partial z}$	$\gamma_z\frac{\partial S}{\partial z}$	$-\gamma_{xzt}\frac{1}{c}\frac{\partial E_x}{\partial z}$	$-\gamma_{yzt}\frac{1}{c}\frac{\partial E_y}{\partial z}$	$\gamma_t\frac{1}{c}\frac{\partial E_z}{\partial z}$	$-\gamma_y\frac{\partial B_x}{\partial z}$	$\gamma_x\frac{\partial B_y}{\partial z}$	$\gamma_{xyz}\frac{\partial B_z}{\partial z}$
$\gamma_t\frac{1}{c}\frac{\partial}{\partial t}$	$\gamma_t\frac{1}{c}\frac{\partial S}{\partial t}$	$\gamma_x\frac{1}{c^2}\frac{\partial E_x}{\partial t}$	$\gamma_y\frac{1}{c^2}\frac{\partial E_y}{\partial t}$	$\gamma_z\frac{1}{c^2}\frac{\partial E_z}{\partial t}$	$\gamma_{yzt}\frac{1}{c}\frac{\partial B_x}{\partial t}$	$-\gamma_{xzt}\frac{1}{c}\frac{\partial B_y}{\partial t}$	$\gamma_{xyt}\frac{1}{c}\frac{\partial B_z}{\partial t}$

$$\gamma_t\frac{1}{c}\left(\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}+\frac{\partial S}{\partial t}\right)=0, \tag{40}$$

$$\gamma_y\gamma_z\gamma_t\frac{1}{c}\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}+\frac{\partial B_x}{\partial t}\right)=0, \tag{41}$$

$$\gamma_x\gamma_z\gamma_t\frac{1}{c}\left(\frac{\partial E_z}{\partial x}-\frac{\partial E_x}{\partial z}-\frac{\partial B_y}{\partial t}\right)=0, \tag{42}$$

$$\gamma_x\gamma_y\gamma_t\frac{1}{c}\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}+\frac{\partial B_z}{\partial t}\right)=0, \tag{43}$$

$$\gamma_x\gamma_y\gamma_z\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}\right)=0. \tag{44}$$

Rearranging all equations from (37) to (44) the following are derived:

$$\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}=\frac{\partial S}{\partial x}+\frac{1}{c^2}\frac{\partial E_x}{\partial t}, \tag{45}$$

$$\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}=\frac{\partial S}{\partial y}+\frac{1}{c^2}\frac{\partial E_y}{\partial t}, \tag{46}$$

$$\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}=\frac{\partial S}{\partial z}+\frac{1}{c^2}\frac{\partial E_z}{\partial t}, \tag{47}$$

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}=-\frac{\partial S}{\partial t}, \tag{48}$$

$$\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=-\frac{\partial B_x}{\partial t}, \tag{49}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t}, \tag{50}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t}, \tag{51}$$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0, \tag{52}$$

which are coincident with Maxwell's equations if

$$\frac{\partial S}{\partial x} = \mu_0 J_{ex} = \mu_0 \frac{\partial q}{\partial y \partial z \partial t} = \mu_0 \frac{\partial q \partial x}{\partial x \partial y \partial z \partial t} = \mu_0 \rho v_x, \tag{53}$$

$$\frac{\partial S}{\partial y} = \mu_0 J_{ey} = \mu_0 \frac{\partial q}{\partial x \partial z \partial t} = \mu_0 \frac{\partial q \partial y}{\partial x \partial y \partial z \partial t} = \mu_0 \rho v_y, \tag{54}$$

$$\frac{\partial S}{\partial z} = \mu_0 J_{ez} = \mu_0 \frac{\partial q}{\partial x \partial y \partial t} = \mu_0 \frac{\partial q \partial z}{\partial x \partial y \partial z \partial t} = \mu_0 \rho v_z, \tag{55}$$

$$\frac{1}{c}\frac{\partial S}{\partial t} = \mu_0 J_{et} = -\mu_0 c \frac{\partial q}{\partial x \partial y \partial z} = -\mu_0 c \rho, \tag{56}$$

where ∂q is the differential of a generic charge [4,17]. Equation (56) can be also written as

$$\frac{\partial S}{\partial t} = c\mu_0 J_{et} = -\mu_0 c^2 \frac{\partial q}{\partial x \partial y \partial z} = -\mu_0 c^2 \rho = -\frac{\rho}{\epsilon_0}. \tag{57}$$

By taking into account (53)–(56), the following relation holds for the current density field,

$$\frac{1}{\mu_0}\boldsymbol{\partial} S = \frac{1}{\mu_0}\left(\gamma_x \frac{\partial S}{\partial x} + \gamma_y \frac{\partial S}{\partial y} + \gamma_z \frac{\partial S}{\partial z} + \gamma_t \frac{1}{c}\frac{\partial S}{\partial t}\right) = \boldsymbol{J}_{\square e}, \tag{58}$$

where

$$\begin{aligned}\boldsymbol{J}_{\square e} &= \gamma_x J_{ex} + \gamma_y J_{ey} + \gamma_z J_{ez} + \gamma_t J_{et} = \gamma_x J_{ex} + \gamma_y J_{ey} + \gamma_z J_{ez} - \gamma_t c\rho \\ &= \boldsymbol{J}_{\triangle} - \gamma_t c\rho = \rho\left(\boldsymbol{v}_{\triangle} - \gamma_t c\right)\end{aligned} \tag{59}$$

is the four-current vector,

$$\boldsymbol{v}_{\square} = \gamma_x v_x + \gamma_y v_y + \gamma_z v_z - \gamma_t c = \boldsymbol{v}_{\triangle} - \gamma_t c \tag{60}$$

is a four-velocity vector and $\boldsymbol{v}_{\triangle}$ is the speed in the ordinary space.

In this formulation the partial derivatives of the scalar field S with respect to time and space coordinates can be interpreted as charge density and current density, respectively. As a matter of fact (45)–(47) represent the spatial components of the Faraday–Neumann–Maxwell–Lenz law, i.e.

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}_{\vartriangle} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}, \tag{61}$$

where $\boldsymbol{J}_{\vartriangle} = \gamma_x J_{ex} + \gamma_y J_{ey} + \gamma_z J_{ez}$ is the three-component vector of current density, (48) is the Gauss's law for the electric field

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}, \tag{62}$$

(49)–(51) represent the spatial components of Ampere's law

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{63}$$

and (52) the Gauss's law for the flux density field

$$\nabla \cdot \boldsymbol{B} = 0. \tag{64}$$

Finally, by applying the $\boldsymbol{\partial}\cdot$ operator to (58) and setting the result to zero, the equation representing the law of electric charge conservation is obtained

$$\frac{1}{\mu_0}\boldsymbol{\partial} \cdot (\boldsymbol{\partial} S) = \boldsymbol{\partial} \cdot \boldsymbol{J}_{\square e} = \frac{\partial J_{ex}}{\partial x} + \frac{\partial J_{ey}}{\partial y} + \frac{\partial J_{ez}}{\partial z} + \frac{\partial \rho}{\partial t} = 0. \tag{65}$$

It is important to note that the wave equation of the scalar field S can be deduced from the charge–current conservation law:

$$\boldsymbol{\partial} \cdot (\boldsymbol{\partial} S) = \boldsymbol{\partial}^2 S = \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 S}{\partial t^2} = \nabla^2 S - \frac{1}{c^2}\frac{\partial^2 S}{\partial t^2} = 0. \tag{66}$$

Now, by applying the time derivative to (66) and remembering (57), the wave equation of the charge field $\rho\left(\boldsymbol{r}_{\square}\right)$ can also be deduced, i.e.

$$\frac{\partial}{\partial t}\left(\boldsymbol{\partial}^2 S\right) = \boldsymbol{\partial}^2 \left(\frac{\partial S}{\partial t}\right) = \boldsymbol{\partial}^2\left(-\mu_0 c^2 \rho\right) = -\mu_0 c^2 \boldsymbol{\partial}^2 \rho = 0, \tag{67}$$

which gives

$$\boldsymbol{\partial}^2 \rho = \nabla^2 \rho - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\rho = 0. \tag{68}$$

Clearly, both (66) and (68) represent, respectively, fields (S and ρ) that must necessary propagate at the speed of light [17,19]. Equation (58) means also that the 4-vector current density field can be derived directly from the scalar field S. The hypothesis of existence of scalar waves has been recently explored at the Oak Ridge laboratories: *"The new theory predicts a new charge-fluctuation-driven scalar wave, having energy but not momentum for zero magnetic and electric fields. The scalar wave can co-exist with a longitudinal-electric wave, having energy and momentum. The new theory in 4-vector form is relativistically covariant. New experimental tests are needed to confirm this theory."* [13].

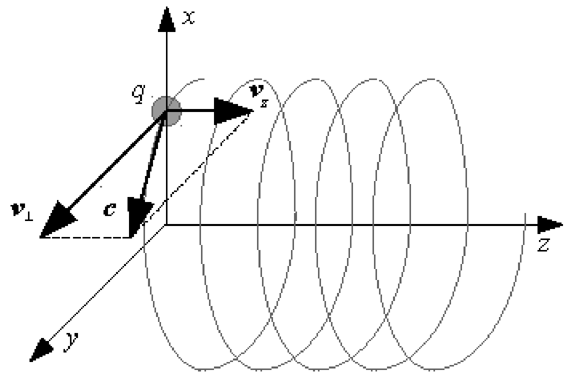

Figure 3. Helical motion of an elementary charge q moving at the speed of light, with $v_z^2 + v_\perp^2 = c^2$.

The proposed reinterpretation of Maxwell's equations in this paper is in agreement with the principle of Occam's razor: the concepts of charge and current density are not inserted "ad hoc" but are deduced from a single more fundamental entity, the four dimensional vector potential field $\boldsymbol{A}_\square\left(\boldsymbol{r}_\square\right) = \boldsymbol{A}_\square\left(x, y, z, t\right)$.

Equation (68) imposes a precise condition on charge dynamics, describing only distributions of charge density moving in vacuum at the speed of light c. At first glance, this result seems to be incompatible with experimental observations, with the usual concepts of charge and current and with the traditional way of working with Maxwell's equations. In fact, with reference to this perspective, a big advantage in using Maxwell's equations is the ability to simply specify both current density and charge density distributions and then see what fields result. Nevertheless, in the model proposed in this paper, the added constraint on the charge and current density seems to imply that one is no longer free to specify charge and current density distributions at will, because this information is indeed included within the electromagnetic four potential $\boldsymbol{A}_\square$.

As will be shown later, we can interpret (68) as a constraint for the definition of models of elementary charges (or particles). This constraint, however, can be removed when considering macroscopic electromagnetic systems or even the dynamics of a single elementary charge at a spatial scale greater than the particle Compton wavelength λ_c and at a time scale greater than the Compton period λ_c/c. In this case static elementary charges can be seen as charge density distributions moving at the speed of light on a closed trajectory but with a zero average speed (this generalization would be consistent with static charge densities, electrets, dielectrics), whereas currents can be considered as an ordered motion of charge density distributions moving with an absolute velocity equal to the speed of light but with an arbitrary average speed lower than c.

As an example, referring to Fig. 3, the electromagnetic effects generated by an elementary charge q, moving at instantaneous speed c in a helical motion of radius $\leq \lambda_\mathrm{c}/2\pi$ with average velocity v_z along the helix axis z and tangential velocity $v_\perp$, can be approximated, on a spatial scale $\gg \lambda_\mathrm{c}$ and a temporal scale $\gg \lambda_\mathrm{c}/c$, to those produced by the same elementary charge q moving at uniform velocity v_z, creating the current density

$$\boldsymbol{J}_z = J_z \gamma_z \approx \frac{q}{\delta x\, \delta y\, \delta z} \frac{\mathrm{d}z}{\mathrm{d}t} \gamma_z = \frac{q}{\delta V} \frac{\mathrm{d}z}{\mathrm{d}t} \gamma_z = \rho v_z \gamma_z = \rho \boldsymbol{v}_z, \tag{69}$$

where $\delta V = \delta x\, \delta y\, \delta z \approx \lambda_\mathrm{c}^3$. In this view and at a macroscopic level the here proposed new interpretation of Maxwell's equations remains compatible with the traditional way of working with them, i.e. by assigning the sources and determining, as a consequence, both the electric and the flux density (magnetic) field.

The new formulation of Maxwell's equations expressed by (35) is quite similar to the Dirac–Hestenes equation for

$m = 0$ (Weyl equation). In all cases the solution is a *spinor* field. A spinor is a mathematical object that in space–time algebra is simply a multivector of even grade components. The motion of a massless charge that moves at speed of light can be described using a composition of a rotation in the $\gamma_x\gamma_y$ plane followed by a scaled hyperbolic rotation in the $\gamma_z\gamma_t$ plane and can be encoded in $Cl_{3,1}$ with a single spinor.

At this point the Authors are encouraged by an interesting sentence of P.A.M. Dirac. In fact, in his Nobel lecture [20], held in 1933, Dirac proposed an electron model in which a charge moves at the speed of light: *"It is found that an electron which seems to us to be moving slowly, must actually have a very high frequency oscillatory motion of small amplitude superposed on the regular motion which appears to us. As a result of this oscillatory motion, the velocity of the electron at any time equals the velocity of light."*

4. Properties of the Electromagnetic Field

In this section, the main properties of the electromagnetic field will be presented and discussed by means of $Cl_{3,1}$ Clifford algebra.

4.1. Lorentz force

A very compact and elegant form for the expression of the Lorentz force can be achieved in $Cl_{3,1}$, in terms of a generic charge q moving at a generic speed $\boldsymbol{v}_\square = \boldsymbol{v}_\triangle - \gamma_t c$, extracting from the expression $q\boldsymbol{G}\boldsymbol{v}_\square$ the blades of degree 1, and considering the four-momentum $\boldsymbol{P}_\square$, i.e.

$$\begin{aligned}\left(\frac{\mathrm{d}\boldsymbol{P}_\square}{\mathrm{d}t}\right)_q &= \left\langle q\boldsymbol{G}\boldsymbol{v}_\square\right\rangle_1 = q\left(\boldsymbol{E} - I_\triangle \boldsymbol{v}_\triangle \wedge \boldsymbol{B} - \gamma_t \frac{1}{c}\boldsymbol{v}_\triangle \cdot \boldsymbol{E} + S\boldsymbol{v}_\triangle - \gamma_t c S\right)\\ &= q\left(\boldsymbol{E} + \boldsymbol{v}_\triangle \times \boldsymbol{B} + S\boldsymbol{v}_\triangle\right) - \gamma_t q\left(\frac{1}{c}\boldsymbol{v}_\triangle \cdot \boldsymbol{E} + cS\right),\end{aligned} \tag{70}$$

where $I_\triangle = \gamma_x\gamma_y\gamma_z = -I\gamma_t$ is the unitary volume of the three dimensional space. In the last member of (70) the first term represents the Lorentz force acting on the charge q plus a force acting on the same charge but depending on the scalar field and on the speed and directed along the motion, whereas the last term in γ_t represents the work carried out by the electric and scalar fields in moving the charge along a unitary distance.

In terms of force density (in N m^{-3}) the above expression becomes

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{P}_{\square V}}{\mathrm{d}t} &= \left\langle \rho\boldsymbol{G}\boldsymbol{v}_\square\right\rangle_1 = \rho\boldsymbol{E} - I_\triangle \boldsymbol{J}_\triangle \wedge \boldsymbol{B} - \gamma_t \frac{1}{c}\boldsymbol{J}_\triangle \cdot \boldsymbol{E} - \gamma_t \rho c S\\ &= \rho\boldsymbol{E} + \boldsymbol{J}_\triangle \times \boldsymbol{B} + S\boldsymbol{J}_\triangle - \gamma_t\left(\frac{1}{c}\boldsymbol{J}_\triangle \cdot \boldsymbol{E} + \rho c S\right),\end{aligned} \tag{71}$$

where $\boldsymbol{P}_{\square V}$ is the four-momentum spatial density, $\boldsymbol{J}_\triangle$ is the generic 3-D current density and ρ the spatial charge density. The term $S\boldsymbol{J}_\triangle$ is the contribution, in terms of force per volume, due to the scalar field; this force density has the same direction of the 3-D current density. The last term in (71), with the unitary vector γ_t, represents the work density produced by the electric and scalar fields when moving the spatial charge density ρ along a unitary distance.

4.2. Derivation of Maxwell's equations from Lagrangian density

Maxwell's equations can be derived considering the following Lagrangian density, in form of a composition of a scalar and a pseudoscalar part:

$$\begin{aligned}\boldsymbol{L} &= \frac{1}{2\mu_0}\boldsymbol{\partial A}_{\square}\widetilde{\boldsymbol{\partial A}_{\square}} = \frac{1}{2\mu_0}\boldsymbol{G}\widetilde{\boldsymbol{G}} = \frac{1}{2\mu_0}\left\|\boldsymbol{G}\right\|^2 = \frac{1}{2\mu_0}\left(S+\boldsymbol{F}\right)\left(S-\boldsymbol{F}\right) = \frac{1}{2\mu_0}\left(S^2-\boldsymbol{F}^2\right)\\ &= \frac{1}{2\mu_0}\left(-\frac{E^2}{c^2}+B^2+S^2-\frac{2}{c}I\boldsymbol{E}\cdot\boldsymbol{B}\right),\end{aligned} \tag{72}$$

where, bearing (33) in mind,

$$\boldsymbol{F} = \frac{1}{c}\boldsymbol{E}\gamma_t + I\boldsymbol{B}\gamma_t = \frac{1}{c}\left(\boldsymbol{E}+Ic\boldsymbol{B}\right)\gamma_t \tag{73}$$

is the bivector part of the electromagnetic field and $\sim$ represents the conjugation operator. Expanding (72), and taking equations from (20) to (26) into account, we obtain the Lagrangian density as a function of the derivatives of the electromagnetic four-potential components, i.e.

$$\begin{aligned}\boldsymbol{L} = \frac{1}{2\mu_0}\Bigg\{ &-\left(\frac{\partial A_t}{\partial x}-\frac{1}{c}\frac{\partial A_x}{\partial t}\right)^2-\left(\frac{\partial A_t}{\partial y}-\frac{1}{c}\frac{\partial A_y}{\partial t}\right)^2-\left(\frac{\partial A_t}{\partial z}-\frac{1}{c}\frac{\partial A_z}{\partial t}\right)^2\\ &+\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)^2+\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)^2+\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)^2\\ &+\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}-\frac{1}{c}\frac{\partial A_t}{\partial t}\right)^2\\ &-2I\left[\left(\frac{\partial A_t}{\partial x}-\frac{1}{c}\frac{\partial A_x}{\partial t}\right)\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\left(\frac{\partial A_t}{\partial y}-\frac{1}{c}\frac{\partial A_y}{\partial t}\right)\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+\right.\\ &\left.+\left(\frac{\partial A_t}{\partial z}-\frac{1}{c}\frac{\partial A_z}{\partial t}\right)\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\right]\Bigg\}.\end{aligned} \tag{74}$$

In $Cl_{3,1}$ algebra the Euler–Lagrange equations can be expressed, considering as variables the electromagnetic four-potential components $A_x(x,y,z,t)$, $A_y(x,y,z,t)$, $A_z(x,y,z,t)$ and $A_t(x,y,z,t)$, in the following way:

$$\sum_{j=x,y,z,t}\left(\sum_{i=x,y,z,t}\gamma_i\frac{\partial}{\partial i}\left(\frac{\partial\boldsymbol{L}}{\gamma_i\gamma_j\partial\left(\frac{\partial A_j}{\partial i}\right)}\right)-\frac{\partial\boldsymbol{L}}{\gamma_j\partial A_j}\right)=0, \tag{75}$$

which reduces itself to

$$\sum_{j=x,y,z,t}\left(\sum_{i=x,y,z,t}\gamma_i\frac{\partial}{\partial i}\left(\frac{\partial\boldsymbol{L}}{\gamma_i\gamma_j\partial\left(\frac{\partial A_j}{\partial i}\right)}\right)\right)=0, \tag{76}$$

considering that in this case

$$\sum_{j=x,y,z,t}\left(\frac{\partial\boldsymbol{L}}{\gamma_j\partial A_j}\right)=0, \tag{77}$$

because in (74) only the derivative terms of the four-potential ($\partial A_j/\partial i$) appear. By expanding (76), for example with $j = t$, we achieve, after some trivial calculation steps,

$$-\gamma_t \frac{\partial \boldsymbol{L}}{\partial A_t} = -\gamma_t \frac{\partial}{\partial x} \frac{\partial \boldsymbol{L}}{\partial \left(\frac{\partial A_t}{\partial x}\right)} - \gamma_t \frac{\partial}{\partial y} \frac{\partial \boldsymbol{L}}{\partial \left(\frac{\partial A_t}{\partial y}\right)} - \gamma_t \frac{\partial}{\partial z} \frac{\partial \boldsymbol{L}}{\partial \left(\frac{\partial A_t}{\partial z}\right)} - \gamma_t \frac{\partial}{\partial t} \frac{\partial \boldsymbol{L}}{\partial \left(\frac{\partial A_t}{\partial t}\right)}$$
$$= \gamma_t \frac{1}{\mu_0} \left(\frac{1}{c} \frac{\partial E_x}{\partial x} + I \frac{\partial B_x}{\partial x} + \frac{1}{c} \frac{\partial E_y}{\partial y} + I \frac{\partial B_y}{\partial y} + \frac{1}{c} \frac{\partial E_z}{\partial z} + I \frac{\partial B_z}{\partial z} + \frac{1}{c} \frac{\partial S}{\partial t} \right) = 0, \tag{78}$$

and this equation returns Gauss's laws for the electric field (see Eq. (40)) and for the flux density field (see Eq. (44)), respectively:

$$\gamma_t \frac{1}{c} \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} + \frac{\partial S}{\partial t} \right) = 0,$$

$$\gamma_x \gamma_y \gamma_z \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) = 0.$$

Now, if we expand (76) with $j = x$, we obtain

$$\gamma_x \frac{\partial \boldsymbol{L}}{\partial A_x} = \gamma_x \frac{\partial}{\partial x} \frac{\partial \boldsymbol{L}}{\partial \left(\frac{\partial A_x}{\partial x}\right)} + \gamma_x \frac{\partial}{\partial y} \frac{\partial \boldsymbol{L}}{\partial \left(\frac{\partial A_x}{\partial y}\right)} + \gamma_x \frac{\partial}{\partial z} \frac{\partial \boldsymbol{L}}{\partial \left(\frac{\partial A_x}{\partial z}\right)} + \gamma_x \frac{\partial}{\partial t} \frac{\partial \boldsymbol{L}}{\partial \left(\frac{\partial A_x}{\partial t}\right)}$$
$$= \gamma_x \frac{1}{\mu_0} \left(\frac{\partial S}{\partial x} - \frac{\partial B_z}{\partial y} + \frac{I}{c} \frac{\partial E_z}{\partial y} + \frac{\partial B_y}{\partial z} - \frac{I}{c} \frac{\partial E_y}{\partial z} + \frac{1}{c^2} \frac{\partial E_x}{\partial t} + \frac{I}{c} \frac{\partial B_x}{\partial t} \right) = 0. \tag{79}$$

Equation (79) gives (37) and (41):

$$\gamma_x \left(\frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y} + \frac{\partial S}{\partial x} + \frac{1}{c^2} \frac{\partial E_x}{\partial t} \right) = 0,$$

$$\gamma_y \gamma_z \gamma_t \frac{1}{c} \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + \frac{\partial B_x}{\partial t} \right) = 0.$$

If we carry on the above procedures with $j = y$ and $j = z$ the other remaining components of Maxwell's equation can be determined, i.e (38), (42), (39) and (43):

$$\gamma_y \frac{\partial \boldsymbol{L}}{\partial A_y} = \gamma_y \frac{\partial}{\partial x} \frac{\partial \boldsymbol{L}}{\partial \left(\frac{\partial A_y}{\partial x}\right)} + \gamma_y \frac{\partial}{\partial y} \frac{\partial \boldsymbol{L}}{\partial \left(\frac{\partial A_y}{\partial y}\right)} + \gamma_y \frac{\partial}{\partial z} \frac{\partial \boldsymbol{L}}{\partial \left(\frac{\partial A_y}{\partial z}\right)} + \gamma_y \frac{\partial}{\partial t} \frac{\partial \boldsymbol{L}}{\partial \left(\frac{\partial A_y}{\partial t}\right)}$$
$$= \gamma_y \frac{1}{\mu_0} \left(\frac{\partial B_z}{\partial x} - \frac{I}{c} \frac{\partial E_z}{\partial x} + \frac{\partial S}{\partial y} - \frac{\partial B_x}{\partial z} + \frac{I}{c} \frac{\partial E_x}{\partial z} + \frac{1}{c^2} \frac{\partial E_y}{\partial t} + \frac{I}{c} \frac{\partial B_y}{\partial t} \right) = 0, \tag{80}$$

$$\gamma_z \frac{\partial \boldsymbol{L}}{\partial A_z} = \gamma_z \frac{\partial}{\partial x} \frac{\partial \boldsymbol{L}}{\partial \left(\frac{\partial A_z}{\partial x}\right)} + \gamma_z \frac{\partial}{\partial y} \frac{\partial \boldsymbol{L}}{\partial \left(\frac{\partial A_z}{\partial y}\right)} + \gamma_z \frac{\partial}{\partial z} \frac{\partial \boldsymbol{L}}{\partial \left(\frac{\partial A_z}{\partial z}\right)} + \gamma_z \frac{\partial}{\partial t} \frac{\partial \boldsymbol{L}}{\partial \left(\frac{\partial A_z}{\partial t}\right)}$$
$$= \gamma_z \frac{1}{\mu_0} \left(-\frac{\partial B_y}{\partial x} + \frac{I}{c}\frac{\partial E_y}{\partial x} + \frac{\partial B_x}{\partial y} - \frac{I}{c}\frac{\partial E_x}{\partial y} + \frac{\partial S}{\partial z} + \frac{1}{c^2}\frac{\partial E_z}{\partial t} + \frac{I}{c}\frac{\partial B_z}{\partial t} \right) = 0, \tag{81}$$

that give, as expected, respectively

$$\gamma_y \left(\frac{\partial B_z}{\partial x} + \frac{\partial S}{\partial y} - \frac{\partial B_x}{\partial z} + \frac{1}{c^2}\frac{\partial E_y}{\partial t} \right) = 0,$$

$$\gamma_x\gamma_z\gamma_t \frac{1}{c} \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} - \frac{\partial B_y}{\partial t} \right) = 0,$$

$$\gamma_z \left(\frac{\partial B_x}{\partial y} + \frac{\partial S}{\partial z} - \frac{\partial B_y}{\partial x} + \frac{1}{c^2}\frac{\partial E_z}{\partial t} \right) = 0,$$

$$\gamma_x\gamma_y\gamma_t \frac{1}{c} \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} + \frac{\partial B_z}{\partial t} \right) = 0.$$

By analyzing the above-reported equations, it is possible to reach some conclusions. First of all the Lagrangian density, as defined in (72), can be divided in the sum of two parts

$$\boldsymbol{L} = L_{\text{field}} + L_{\text{int}}. \tag{82}$$

The first part

$$L_{\text{field}} = \frac{1}{2\mu_0} \left(-\frac{E^2}{c^2} + B^2 \right) = -\frac{1}{2\mu_0} \left(\boldsymbol{F} \cdot \boldsymbol{F} \right) \tag{83}$$

represents the "field part" of the Lagrangian density, as known in literature, and the second

$$L_{\text{int}} = \frac{1}{2\mu_0} \left(S^2 - \frac{2}{c} I \boldsymbol{E} \cdot \boldsymbol{B} \right) \tag{84}$$

represents the "interaction term" of the Lagrangian density, that takes the interaction of the electromagnetic field with the sources into account, remembering, in addition, that the derivatives of the scalar field S, with respect to the four dimensional space coordinates x, y, z and t, are bounded respectively to the sources J_{ex}, J_{ey}, J_{ez} and $J_{et} = -c\rho$ (see Eq. (58)). Indeed, by deriving only the interaction terms of the Lagrangian density with respect to the four-potential, i.e. by performing the operation $\partial L_{\text{int}}/\partial A_j$, it is possible to derive the term $\boldsymbol{J}_{\square e} \cdot \boldsymbol{A}_{\square}$. In fact, for the component along γ_t we find

$$
\begin{aligned}
-\gamma_t \frac{\partial L_{\text{int}}}{\partial A_t} &= -\gamma_t \frac{\partial}{\partial x} \frac{\partial L_{\text{int}}}{\partial \left(\frac{\partial A_t}{\partial x}\right)} - \gamma_t \frac{\partial}{\partial y} \frac{\partial L_{\text{int}}}{\partial \left(\frac{\partial A_t}{\partial y}\right)} - \gamma_t \frac{\partial}{\partial z} \frac{\partial L_{\text{int}}}{\partial \left(\frac{\partial A_t}{\partial z}\right)} - \gamma_t \frac{\partial}{\partial t} \frac{\partial L_{\text{int}}}{\partial \left(\frac{\partial A_t}{\partial t}\right)} \\
&= \frac{\gamma_t}{\mu_0} \left(I \frac{\partial B_x}{\partial x} + I \frac{\partial B_y}{\partial y} + I \frac{\partial B_z}{\partial z} + \frac{1}{c} \frac{\partial S}{\partial t} \right) = \frac{\gamma_t}{\mu_0} \left(I \nabla \cdot \boldsymbol{B} + \frac{1}{c} \frac{\partial S}{\partial t} \right) \\
&= \frac{\gamma_t}{\mu_0 c} \frac{\partial S}{\partial t} = \gamma_t J_{et} = -\gamma_t c \rho. \qquad (85)
\end{aligned}
$$

Integration of (85) yields

$$
L_{\text{int}}|_t = \int \frac{\partial L_{\text{int}}}{\partial A_t} \mathrm{d}A_t = \int \frac{1}{\mu_0 c} \frac{\partial S}{\partial t} \mathrm{d}A_t = \frac{1}{\mu_0 c} \frac{\partial S}{\partial t} A_t = -\frac{1}{\mu_0} \mu_0 c \rho A_t = -c \rho A_t = J_{et} A_t. \qquad (86)
$$

For the component along γ_x we find

$$
\begin{aligned}
\gamma_x \frac{\partial L_{\text{int}}}{\partial A_x} &= \gamma_x \frac{\partial}{\partial x} \frac{\partial L_{\text{int}}}{\partial \left(\frac{\partial A_x}{\partial x}\right)} + \gamma_x \frac{\partial}{\partial y} \frac{\partial L_{\text{int}}}{\partial \left(\frac{\partial A_x}{\partial y}\right)} + \gamma_x \frac{\partial}{\partial z} \frac{\partial L_{\text{int}}}{\partial \left(\frac{\partial A_x}{\partial z}\right)} + \gamma_x \frac{\partial}{\partial t} \frac{\partial L_{\text{int}}}{\partial \left(\frac{\partial A_x}{\partial t}\right)} \\
&= \frac{\gamma_x}{\mu_0} \left(\frac{\partial S}{\partial x} + \frac{I}{c} \frac{\partial E_z}{\partial y} - \frac{I}{c} \frac{\partial E_y}{\partial z} + \frac{I}{c} \frac{\partial B_x}{\partial t} \right) = \frac{\gamma_x}{\mu_0} \frac{\partial S}{\partial x} = \gamma_x J_{ex}. \qquad (87)
\end{aligned}
$$

Integration of (87) yields

$$
L_{\text{int}}|_x = \int \left(\frac{\partial L_{\text{int}}}{\partial A_x} \right) \mathrm{d}A_x = \int \frac{1}{\mu_0} \frac{\partial S}{\partial x} \mathrm{d}A_x = \frac{1}{\mu_0} \frac{\partial S}{\partial x} A_x = J_{ex} A_x. \qquad (88)
$$

The same procedure is clearly valid also for the components in γ_y and γ_z. Finally, by integration of (77), we get the Lagrangian density interaction term as

$$
L_{\text{int}} = \sum_{j=x,y,z,t} \int \left(\frac{\partial L_{\text{int}}}{\partial A_j} \right) \mathrm{d}A_j = J_{ex} A_x + J_{ey} A_y + J_{ez} A_z - c \rho A_t = \boldsymbol{J}_{\Box e} \cdot \boldsymbol{A}_{\Box}, \qquad (89)
$$

which is the usual "source" term that is added in traditional Lagrangian theory for classical electricity and magnetism in order to obtain the complete set of Maxwell's equations [6-8,17]. The scalar product $\boldsymbol{J}_{\Box e} \cdot \boldsymbol{A}_{\Box}$ has a dimension of energy per volume (J m^{-3}); in particular, the contribution of the spatial components of vectors $\boldsymbol{J}_{\Box e}$ and $\boldsymbol{A}_{\Box}$ (the scalar product $\boldsymbol{J}_{\triangle} \cdot \boldsymbol{A}_{\triangle}$) can be considered as the specific "kinetic" energy of the electromagnetic field, whereas the term $J_{et} A_t = -c \rho A_t$ the "potential" energy. By virtue of (84), (89) becomes

$$
L_{\text{int}} = \frac{1}{2\mu_0} \left(S^2 - \frac{2}{c} I \boldsymbol{E} \cdot \boldsymbol{B} \right) = \boldsymbol{J}_{\Box e} \cdot \boldsymbol{A}_{\Box}. \qquad (90)
$$

The pseudoscalar term $2/cI\boldsymbol{E} \cdot \boldsymbol{B}$ is clearly null as the electric and the magnetic flux density fields are always orthogonal with respect to each other: indeed, this term contains information about (63). A direct consequence of (90) is, therefore, the following relation between the scalar field, the electromagnetic four-potential and the four-current density:

$$
S^2 = 2\mu_0 \boldsymbol{J}_{\Box e} \cdot \boldsymbol{A}_{\Box}. \qquad (91)
$$

By inspection of (85) and (87), and generalizing, it is possible to define the four-current vector $\boldsymbol{J}_{\square e}$ from the interaction Lagrangian term:

$$\sum_{j=x,y,z,t}\left(\frac{\partial L_{\text{int}}}{\gamma_j \partial A_j}\right) = \gamma_x \frac{\partial L_{\text{int}}}{\partial A_x} + \gamma_y \frac{\partial L_{\text{int}}}{\partial A_y} + \gamma_z \frac{\partial L_{\text{int}}}{\partial A_z} - \gamma_t \frac{\partial L_{\text{int}}}{\partial A_t}$$
$$= \gamma_x J_{ex} + \gamma_y J_{ey} + \gamma_z J_{zx} + \gamma_t J_{et} = \boldsymbol{J}_{\square e}. \quad (92)$$

and, again, by virtue of the Noether's theorem, the law of current and charge conservation

$$\boldsymbol{\partial} \cdot \left[\sum_{j=x,y,z,t}\left(\frac{\partial L_{\text{int}}}{\gamma_j \partial A_j}\right)\right] = \boldsymbol{\partial} \cdot \boldsymbol{J}_{\square e} = 0, \quad (93)$$

which returns, consequently, the wave equations (66) and (68), respectively.

As can be seen the definition of the electromagnetic field $\boldsymbol{G}$ is complete and it includes itself the information of both action and interaction, without the need of any additional term: this is in full accordance with the principle of Occam's razor.

Thanks to the $Cl_{3,1}$ Clifford algebra the Euler–Lagrange equations can be conveniently defined in in a very compact form:

$$\boldsymbol{\partial}\left(\frac{\partial L}{\partial\left(\boldsymbol{\partial} \wedge \boldsymbol{A}_{\square}\right)}\right) - \frac{\partial L}{\partial \boldsymbol{A}_{\square}} = \boldsymbol{\partial}\left(\frac{\partial L}{\partial \boldsymbol{F}}\right) - \frac{\partial L}{\partial \boldsymbol{A}_{\square}} = 0, \quad (94)$$

where, now, the scalar Lagrangian density is

$$L = L_{\text{field}} + L_{\text{int}} = -\frac{1}{2\mu_0}\boldsymbol{F} \cdot \boldsymbol{F} + \boldsymbol{J}_{\square e} \cdot \boldsymbol{A}_{\square}. \quad (95)$$

Substituting (95) in (94) one can achieve directly Maxwell's equations in $Cl_{3,1}$ in the form shown in the previous sections (see Eq. (35)), i.e.

$$\boldsymbol{\partial}\left(\frac{\partial\left(-\frac{1}{2\mu_0}\boldsymbol{F} \cdot \boldsymbol{F} + \boldsymbol{J}_{\square e} \cdot \boldsymbol{A}_{\square}\right)}{\partial \boldsymbol{F}}\right) - \frac{\partial\left(-\frac{1}{2\mu_0}\boldsymbol{F} \cdot \boldsymbol{F} + \boldsymbol{J}_{\square e} \cdot \boldsymbol{A}_{\square}\right)}{\partial \boldsymbol{A}_{\square}} = -\frac{1}{\mu_0}\boldsymbol{\partial}\boldsymbol{F} - \boldsymbol{J}_{\square e} = 0, \quad (96)$$

which yields

$$\boldsymbol{\partial}\boldsymbol{F} + \mu_0 \boldsymbol{J}_{\square e} = \boldsymbol{\partial}\boldsymbol{F} + \boldsymbol{\partial} S = \boldsymbol{\partial}\left(\boldsymbol{F} + S\right) = \boldsymbol{\partial}\boldsymbol{G} = 0. \quad (97)$$

4.3. Energy of the electromagnetic field

The energy density multivector of the field $\boldsymbol{G}$ is given by

$$\begin{aligned}\boldsymbol{w} &= \frac{1}{2\mu_0}\left[S+\left(\frac{\boldsymbol{E}}{c}+I\boldsymbol{B}\right)\gamma_t\right]\left[S+\left(\frac{\boldsymbol{E}}{c}+I\boldsymbol{B}\right)^{\dagger}\gamma_t\right] \\ &= \frac{1}{2\mu_0}\left[S+\left(\frac{\boldsymbol{E}}{c}+I\boldsymbol{B}\right)\gamma_t\right]\left[S+\left(\frac{\boldsymbol{E}}{c}-I\boldsymbol{B}\right)\gamma_t\right] \\ &= \frac{S^2}{2\mu_0}+\frac{\epsilon_0\boldsymbol{E}^2}{2}+\frac{\boldsymbol{B}^2}{2\mu_0}-\frac{1}{c\mu_0}I\boldsymbol{E}\wedge\boldsymbol{B}+\frac{1}{c\mu_0}S\boldsymbol{E}\gamma_t \\ &= \frac{S^2}{2\mu_0}+\frac{\epsilon_0\boldsymbol{E}^2}{2}+\frac{\boldsymbol{B}^2}{2\mu_0}+\frac{1}{c\mu_0}\left(\boldsymbol{E}\times\boldsymbol{B}+S\boldsymbol{E}\right)\gamma_t \\ &= w_{\mathrm{s}}+w_{\mathrm{e}}+w_{\mathrm{m}}+\frac{1}{c}\cdot\mathfrak{S}\gamma_t,\end{aligned} \tag{98}$$

where

$$w_{\mathrm{s}} = \frac{S^2}{2\mu_0} = \boldsymbol{J}_{\square e}\cdot\boldsymbol{A}_{\square},\quad w_{\mathrm{e}} = \frac{\epsilon_0\boldsymbol{E}^2}{2}\quad \text{and}\quad w_{\mathrm{m}} = \frac{\boldsymbol{B}^2}{2\mu_0}$$

are the specific energies of the scalar, the electric and the magnetic flux density fields, respectively, † is the reversion operator, whereas

$$\mathfrak{S} = \frac{1}{\mu_0}\left(\boldsymbol{E}\times\boldsymbol{B}+S\boldsymbol{E}\right) \tag{99}$$

is the generalized Poynting vector [10,11]. Beside the usual term $\boldsymbol{E}\times\boldsymbol{B}$ here a new energy term appears, namely $S\boldsymbol{E}$, which is associated to a longitudinal scalar wave [11] and that is not further investigated in the present work.

4.4. Electrostatic field and vector potential

In the case of non-time-varying potential $\partial A_t/\partial t = 0$ the scalar field S becomes the divergence of the classical 3-D vector potential $\boldsymbol{A}_{\triangle}$:

$$S = \nabla\cdot\boldsymbol{A}_{\triangle}.$$

The time derivative of both sides gives

$$\frac{\partial S}{\partial t} = \frac{\partial(\nabla\cdot\boldsymbol{A}_{\triangle})}{\partial t} = \nabla\cdot\frac{\partial\boldsymbol{A}_{\triangle}}{\partial t}.$$

Considering that $\partial\boldsymbol{A}_{\triangle}/\partial t = -\boldsymbol{E}$ (see Eqs. (21)–(23)) and that $\partial S/\partial t = -\rho/\epsilon_0$ we rediscover Gauss's law,

$$\nabla\cdot\boldsymbol{E} = \frac{\rho}{\epsilon_0},$$

showing that even the electrostatic field may be seen as generated from time derivatives of a vector potential field!

4.5. Electric charge, antimatter and time direction

Feynman proposed to interpret the positron (a particle which is identical to the electron but with a positive charge) as an electron traveling back in time. Such an interpretation seems to be perfectly compatible with the definition of electric charge density given in (56):

$$\frac{\partial S}{\partial t} = \frac{-\rho}{\epsilon_0}. \tag{100}$$

By multiplying both sides of (100) by –1 we obtain

$$-\frac{\partial S}{\partial t} = \frac{\rho}{\epsilon_0}$$

or, equivalently

$$\frac{\partial S}{\partial(-t)} = \frac{\rho}{\epsilon_0}. \tag{101}$$

The positron traveling back in time is represented in the annihilation reaction diagram proposed by Feynman and shown in Fig. 4 [21].

"I did not take the idea that all the electrons were the same one from him as seriously as I took the observation that positrons could simply be represented as electrons going from the future to the past in a back section of their world lines" [22].

4.6. Magnetic charges and currents

Starting from an hypothetical eight component "vector potential" that includes the four pseudovectors (trivectors) $\boldsymbol{T}$ of space–time algebra, symmetrical Maxwell's equations emerge. This new set of equations now include the magnetic charge and magnetic current densities that are the time and spatial derivatives of a pseudoscalar field P. By considering (9) and the four pseudovectors defined as

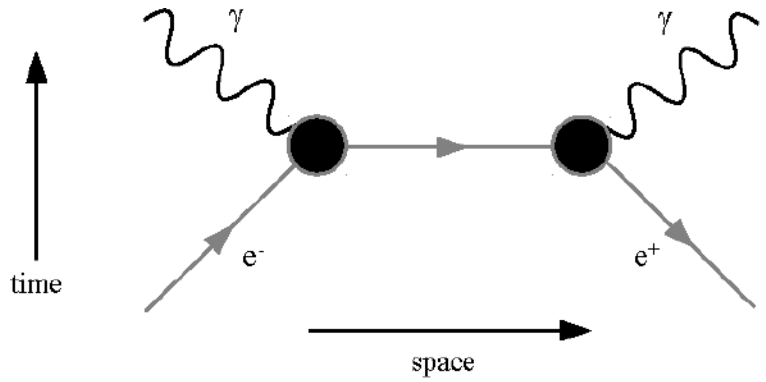

Figure 4. Feynman's diagram of proton–electron annihilation.

$$\boldsymbol{T} = \gamma_y\gamma_z\gamma_t T_x + \gamma_x\gamma_z\gamma_t T_y + \gamma_x\gamma_y\gamma_t T_z + \gamma_x\gamma_y\gamma_z T_t, \tag{102}$$

a new vector potential can be defined as

$$\boldsymbol{A'} = \gamma_x A_x + \gamma_y A_y + \gamma_z A_z + \gamma_t A_t + \gamma_y\gamma_z\gamma_t T_x + \gamma_x\gamma_z\gamma_t T_y + \gamma_x\gamma_y\gamma_t T_z + \gamma_x\gamma_y\gamma_z T_t \tag{103}$$

from which we obtain

$$\boldsymbol{\partial}(\boldsymbol{A}') = \boldsymbol{\partial}\left(\boldsymbol{A}_{\square} + \boldsymbol{T}\right) = S + \boldsymbol{F} + P. \tag{104}$$

Using SI units and following the same procedure as shown in Section 3 we can write:

$$S = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} - \frac{1}{c}\frac{\partial A_t}{\partial t},$$

$$\gamma_x\gamma_t\frac{1}{c}E_x = \gamma_x\gamma_t\left(\frac{\partial A_t}{\partial x} - \frac{\partial T_z}{\partial y} - \frac{\partial T_y}{\partial z} - \frac{1}{c}\frac{\partial A_x}{\partial t}\right),$$

$$\gamma_y\gamma_t\frac{1}{c}E_y = \gamma_y\gamma_t\left(\frac{\partial T_z}{\partial x} + \frac{\partial A_t}{\partial y} - \frac{\partial T_x}{\partial z} - \frac{1}{c}\frac{\partial A_y}{\partial t}\right),$$

$$\gamma_z\gamma_t\frac{1}{c}E_z = \gamma_z\gamma_t\left(\frac{\partial T_y}{\partial x} + \frac{\partial T_x}{\partial y} + \frac{\partial A_t}{\partial z} - \frac{1}{c}\frac{\partial A_z}{\partial t}\right),$$

$$\gamma_x\gamma_y\gamma_z\gamma_t P = \gamma_x\gamma_y\gamma_z\gamma_t\left(\frac{\partial T_x}{\partial x} - \frac{\partial T_y}{\partial y} + \frac{\partial T_z}{\partial z} - \frac{1}{c}\frac{\partial T_t}{\partial t}\right),$$

$$\gamma_y\gamma_z B_x = \gamma_y\gamma_z\left(\frac{\partial T_t}{\partial x} + \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} - \frac{1}{c}\frac{\partial T_x}{\partial t}\right),$$

$$\gamma_x\gamma_z B_y = \gamma_x\gamma_z\left(-\frac{\partial A_z}{\partial x} + \frac{\partial T_t}{\partial y} + \frac{\partial A_x}{\partial z} + \frac{1}{c}\frac{\partial T_y}{\partial t}\right),$$

$$\gamma_x\gamma_y B_z = \gamma_x\gamma_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} + \frac{\partial T_t}{\partial z} - \frac{1}{c}\frac{\partial T_z}{\partial t}\right).$$

By applying again the $\boldsymbol{\partial}$ operator to (104) and equating to zero:

$$\boldsymbol{\partial}^2\boldsymbol{A}' = \boldsymbol{\partial}\left(S + \boldsymbol{F} + P\right) = \boldsymbol{0}. \tag{105}$$

Here

$$\boldsymbol{\partial F} = -\boldsymbol{\partial} S - \boldsymbol{\partial} P = \mu_0 \boldsymbol{J}_{\Box e} + \frac{1}{\epsilon_0}\boldsymbol{J}_{\Box \mathrm{m}}, \tag{106}$$

where $\boldsymbol{J}_{\Box e}$ is the four-current as defined in (58),

$$\boldsymbol{J}_{\Box \mathrm{m}} = \gamma_x J_{\mathrm{m}x} + \gamma_y J_{\mathrm{m}y} + \gamma_z J_{\mathrm{m}z} + \gamma_t J_{\mathrm{m}t} = \gamma_x J_{\mathrm{m}x} + \gamma_y J_{\mathrm{m}y} + \gamma_z J_{\mathrm{m}z} - \gamma_t \frac{1}{c}\rho_{\mathrm{m}}$$

is the magnetic four-current vector and ρ_{m} the magnetic charge. By carrying out all calculation in (105) the following set of equations is obtained:

$$\gamma_x \left(\frac{\partial S}{\partial x} - \frac{\partial B_z}{\partial y} + \frac{\partial B_y}{\partial z} + \frac{1}{c^2}\frac{\partial E_x}{\partial t} \right) = 0,$$

$$\gamma_y \left(\frac{\partial B_z}{\partial x} + \frac{\partial S}{\partial y} - \frac{\partial B_x}{\partial z} + \frac{1}{c^2}\frac{\partial E_y}{\partial t} \right) = 0,$$

$$\gamma_z \left(-\frac{\partial B_y}{\partial x} + \frac{\partial B_x}{\partial y} + \frac{\partial S}{\partial z} + \frac{1}{c^2}\frac{\partial E_z}{\partial t} \right) = 0,$$

$$\gamma_t \frac{1}{c} \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} + \frac{\partial S}{\partial t} \right) = 0,$$

$$\gamma_y\gamma_z\gamma_t \frac{1}{c} \left(\frac{\partial P}{\partial x} + \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + \frac{\partial B_x}{\partial t} \right) = 0,$$

$$\gamma_x\gamma_z\gamma_t \frac{1}{c} \left(\frac{\partial E_z}{\partial x} - \frac{\partial P}{\partial y} - \frac{\partial E_x}{\partial z} - \frac{\partial B_y}{\partial t} \right) = 0,$$

$$\gamma_x\gamma_y\gamma_t \frac{1}{c} \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} + \frac{\partial P}{\partial z} + \frac{\partial B_z}{\partial t} \right) = 0,$$

$$\gamma_x\gamma_y\gamma_z \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} + \frac{\partial P}{\partial t} \right) = 0.$$

This set of equations represents the symmetrical Maxwell's equations considering the hypothesis (never confirmed up until now by experiments) of existing magnetic currents and charges.

5. Conclusions

Simplicity is an important and concrete value in scientific research. Connections between very different concepts in physics can be evidenced if we use the language of geometric algebra. The application of Occam's Razor principle to Maxwell's equations highlights some essential concepts:

(1) First of all, Clifford algebra is by far the most appropriate, simple and intuitive mathematical language for encoding in general the laws of physics and in particular the laws of electromagnetism.
(2) A scalar field derives from the definition of "harmonic" electromagnetic four-potential and this scalar field becomes the source of charges and currents.
(3) The charge density derived from the scalar field follows the wave equation with a propagation speed equal to the speed of light.
(4) The Feynman model of the positron, seen as an electron traveling back in time, seems to be perfectly compatible with the definition of electric charge density as the time derivative of a scalar field.

In particular, the important element emerging from the present paper is that (68) imposes a precise condition on charge dynamics, describing distributions of charge density moving in vacuum at the speed of light. Indeed, van Vlaenderen found the same condition on charge dynamics but with the difference that he considers both the conditions $\boldsymbol{E} = 0$ and $\boldsymbol{B} = 0$ at the same time (no electromagnetic field) concluding that *"a scalar field S can be induced by a dynamic charge/current distribution"* [11].

In the model proposed here, the added constraint on the charge and current density seems to imply that one is no longer free to specify charge and current density distributions at will, because this information is indeed included within the definition of the four-potential $\boldsymbol{A}_{\square}$. However, this constraint can be removed when considering macroscopic electromagnetic systems or even the dynamics of a single elementary charge at a spatial scale greater than the particle Compton wavelength λ_c and at a time scale greater than the Compton period λ_c/c. In this case static elementary charges can be visualized as charge density distributions moving at the speed of light on a closed trajectory but with a zero average speed (this generalization would be consistent with static charge densities, electrets, dielectrics), whereas currents can be considered as an ordered motion of charge density distributions moving with an absolute velocity equal to the speed of light but with an arbitrary absolute average speed lower than c. This observation favors a pure electromagnetic model of elementary particles based on a particular Zitterbewegung interpretation of quantum mechanics [23,24]. Therefore, the free electron, and perhaps all other elementary charged particles, can be viewed as a charge distribution that rotates at the speed of light along a circumference whose length is equal to its Compton wavelength [25].

Finally, a Lagrangian density equal to the square module of the seven component electromagnetic field reveals an energy density formula for both fields and currents. Moreover, it has been demonstrated that Maxwell's equations can be explicitly derived in a simple way directly from the Lagrangian density of the electromagnetic field with the help of Clifford algebra. An interesting consequence is also that the specific energy of the scalar field is deeply connected to the interaction term of the Lagrangian density and, therefore, both to the electromagnetic four-potential and the four-current density.

It is our opinion that the *Zitterbewegung* interpretation of quantum mechanics may give an important contribution for understanding the structure of ultradense hydrogen and the origin of anomalous heat in some metal–hydrogen systems. A *Zitterbewegung* electron model and a preliminary hypothesis for the structure of ultradense deuterium will be treated more deeply in a paper written by the authors, entitled "The Electron and Occam's Razor", *J. Condensed Matter Nucl. Sci.* **25** (2017).

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