Research Article

Advance on Electron Deep Orbits of the Hydrogen Atom

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Abstract

In the previous works, we discussed arguments for and against the deep orbits, as exemplified in published solutions. So we considered the works of Maly and Va'vra on the topic, the most complete solution available and one showing an infinite family of EDO solutions. In particular, we deeply analyzed their second of these papers, where they consider a finite nucleus and look for solutions with a Coulomb potential modified inside the nucleus. In the present paper, we quickly recall our analysis, verification, and extension of their results. Moreover, we answer to a recent criticism that the EDOs would represent negative energy states and therefore would not qualify as an answer to the questions posed by Cold Fusion results. We can prove, by means of a simple algebraic argument based on the solution process, that, while at the transition region, the energy of the EDOs are positive. Next, we deepen the essential role of Special Relativity as source of the EDOs, which we discussed in previous papers. But the central topic of our present study is an initial analysis of the magnetic interactions near the nucleus, with the aim of solving important physical questions: do the EDOs satisfy the Heisenberg Uncertainty relation (HUR)? Are the orbits stable? So, we examine some works related to the Vigier–Barut Model, with potentials including magnetic coupling. We also carried out approximate computations to evaluate the strength of these interactions and the possibilities of their answering some of our questions. As a first result, we can expect the HUR to be respected by EDOs, due to the high energies of the magnetic interactions near the nucleus. Present computations for stability do not yet give a plain result; we need further studies and tools based on QED to face the complexity of the near-nuclear region. For the creation of EDOs, we outline a possibility based on magnetic coupling.

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1. Introduction

The question of the existence of electron deep orbits (EDO) for the hydrogen atom has led to a number of works and debates.

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In previous works [1,2] we answered to main criticism found in the literature, which are of a mathematical nature. Also, in this paper, we only list them with short summaries of our solutions and of the origin of the EDOs as solutions of relativistic equations for atomic H (essentially the relativistic Schrödinger equation – also called the Klein–Gordon equation – and the Dirac equation for a single particle in a potential).

Recently, we faced a new and unexpected argument against the deep orbits, concerning the sign of their energy levels. As a consequence, we present here a straightforward algebraic reasoning, based on known methods to solve the radial Dirac equation for atomic H, which allows us to show that deep orbits have positive energy.

With this perspective, it is interesting to see how Special Relativity plays a central role in the theoretical genesis of the EDOs, when considering only the Coulomb potential. Regarding this subject, we open a little discussion about the singular solutions in a non-relativistic context. One can ask what is precisely the element, in Special Relativity, that is the source of EDOs. For the present, and from a more precise analysis [3], we can think that a quadratic relation involving the energy E, eigenvalue of the Hamiltonian, and the potential energy, inserted into the equation, plays a dominant role.

However, as the heart of our present paper, we now analyze the role of various kinds of magnetic interactions inside the atom, for the EDOs. For example: is the Heisenberg uncertainty relation satisfied by electrons so confined near the nucleus? That is effectively a very important question, of physical nature and touching an essential pillar of the Quantum Mechanics. So, we recall most known magnetic interactions in a simple atom H, and examine works explicitly including several magnetic potentials in the Hamiltonian of the atom. Do not forget the Dirac equation includes (in a hidden way) the spin-orbit interaction for electron spin, but no interaction with the nuclear spin; and the Schrödinger equation does not include any spin considerations (Note: While it is generally accepted that the Schrödinger equation does not consider spin, one can find in the literature [4] arguments suggesting that it represents a particle in a spin eigenstate.) Our first study on the magnetic interactions and on the works based on the Vigier-Barut model, lead us to make some correlations with other works and results. After developing this subject, we made several approximate computations, in order to estimate the energies of the interactions, which, as expected, are very high when approaching the nucleus.

Partial results given at the end of this study are hopeful, especially about the HUR.

2. Deep Orbits Obtained as Solutions of Relativistic Equations for Atom H

In this section, we quickly recall our analysis, verification and extension of works about EDOs obtained by relativistic quantum equations, and exposed in previous papers [5,6] in a developed way. While looking for the solutions of the radial equations by an analytic method, there is a parameter generally denoted s that occurs during the process. This parameter appears in an exponential term of form ρ^s included in the used ansatz, and s has to satisfy a quadratic relation on the coefficients of series included in the ansatz.

In the case of the relativistic Schrödinger equation, this quadratic relation for s is the following: $s(s+1) + \alpha^2 - l(l+1) = 0$, where l is the usual angular number and α the coupling constant. This equation admits two roots: $s = -1/2 \pm [(l + 1/2)^2 - \alpha^2]^{1/2}$. For the Dirac equation, the condition on s is the following: $s^2 - k^2 + \alpha^2 = 0$, where k is the specific Dirac angular number.

This equation also admits two roots: $s = \pm(k^2 - \alpha^2)^{1/2}$. For both cases of relativistic equations, when one takes the "plus" sign for s, one obtains the usual "regular" solutions, whereas with the "minus" sign one obtains the so-called "anomalous" or "singular" solutions.

We recall here the expressions of energy levels of these singular solutions. For the Schrödinger, it reads:

$$E_{\mathrm{S}} = mc^2 \left[1 + \frac{\alpha^2}{\left(n' + \frac{1}{2} - \left[\left(l + \frac{1}{2} \right)^2 - \alpha^2 \right]^{\frac{1}{2}} \right)} \right]^{-1/2}, \qquad (1)$$

while for the Dirac equation, we have:

$$E_{\mathrm{D}} = mc^2 \left[1 + \frac{\alpha^2}{\left(n' - \sqrt{(k^2 - \alpha^2)} \right)^2} \right]^{-1/2}. \qquad (2)$$

In previous works, we noted that all energy values given by E_{S} or E_{D} do not correspond to deep orbits. In fact, the deep orbits are obtained only when the angular number (l or $|k|$) is equal to the radial number n'. Note that for the Dirac equation, one takes the absolute value of k, because k can be positive or negative, but only $|k|$ is involved in the computation of the E_{D}.

Moreover, the EDOs solutions have the following properties:

- The total energy E_{S} (or E_{D}) $\ll mc^2$, which implies the binding energy $|\mathrm{BE}|$ is near the rest mass energy, more precisely, for H, $|\mathrm{BE}| > 509$ keV for the E_{D} solutions, and $|\mathrm{BE}| > 507$ keV for the E_{S} solutions. This difference is only due to the fact that the Schrödinger equation admits an additional solution, for $n' = l = 0$, whereas $n' = k = 0$ is not possible for the Dirac equation, since k cannot be equal to 0.
- When n' increases, for E_{S} or E_{D}, the absolute value $|\mathrm{BE}|$ of the binding energy increases.
- Moreover, the condition $|k| = n'$ (or $l = n'$) separates the positive- and negative-energy solutions. More precisely:
- When $|k|$(or l) $> n'$, E_{S} or E_{D} correspond to "negative energy" states.
- When $|k|$ (or l) $< n'$, the energies are positive and their values are near those of regular solutions for $s > 0$. We called them "pseudo-regular" solutions.

The question of the sign of the energy for the EDO is analyzed in a more developed way in Section 3.

2.1. Possible solutions to the arguments against the EDO states

Here, we only summarize our previous studies, about answers to mathematical arguments against the EDOs.

In the literature, there was essentially the following criticism:

(1) The wavefunction has a singular point at the origin
(2) The wavefunction cannot be square integrable
(3) The orthogonality of eigenvectors cannot be satisfied
(4) There is a paradoxical relationship between the coupling constant α and the binding energy.

For the point #4, we concluded the result in question is obtained in the context of an ill-defined system. For the other points, we reported the deep orbit solution Klein–Gordon equation, given in [7] in the case $l = 0$ is square integrable, and according to works [8], it satisfies the orthogonality conditions and boundary conditions. Nevertheless, in the other cases, and in particular the Dirac equation, the difficulties #1 to #3 are related to the singularity of the Coulomb potential.

Then, we analyzed the works of [9], where the authors remove the singularity by taking into account the nucleus is not a point, and look for solutions with a modified potential inside the nucleus. We summarize the computation process used:

- To choose a radius R_0, so-called "matching radius", delimiting two spatial domains: an "outside" one, where the potential is expressed by the usual Coulomb potential, and an "inside", where a chosen potential, different of the Coulomb potential has no longer a singular point at $r = 0$. Of course, the outside Coulomb potential and the inside potential have to be connected at the matching radius, while satisfying continuity conditions.
- To solve the system of radial Dirac equations for the "outside potential", i.e. the Coulomb potential, which gives outside solutions consisting of two components.
- To solve the system for the chosen inside potential, which gives the "inside solution" consisting of two components.
- To satisfy continuity conditions while connecting the respective components of inside and outside solutions.

In fact, during our analysis, we had to look for a more complex ansatz [5] for the inside solutions to satisfy more complete continuity conditions at the matching radius. Next we computed the electron probability density for the wavefunctions, and values of the mean radius $\langle r \rangle$ of the deep orbits, for increasing values of $k(= n')$.

2.2. Results on EDOs obtained as solutions of Dirac equation with a corrected potential at the origin

Numerical computations were computed by choosing the matching radius $R_0 = 1.2$ F for atomic H. For the inside solutions, the ansatz includes polynomials of degree 6. The outside solutions are those indicated in [9], i.e. they are built with hyper-geometrical series determined by an analytic method as indicated in [10]. We made computations of the mean radius for increasing values of k, to show the progression of the $\langle r \rangle$ as a function of k. So, for k respectively equal to 1,2,3,10,20, we respectively obtain $\langle r \rangle = 6.62, 1.65, 1.39, 1.22, 1.2$ in F. These results confirm and extend those of [9]. Moreover they show $\langle r \rangle$ decreases when k increases: it is consistent with $|\text{BE}|$ increasing when $k(= n')$ increases.

Further computations were carried out, while changing the values of some parameters: e.g. the precision of our ansatz, the form of the nuclear potential, the chosen matching radius R_0. Note that the choice of the matching radius has to depend on the charge radius of the nucleus: for a greater Z, one has to take a greater value for R_0. With these conditions, we observed the values of $\langle r \rangle$ are nearly independent of the parameters except for the value of the matching radius R_0.

Next, we considered two related facts:

- On the one hand, we have some lack of dependence of the solutions on the nuclear electrostatic potential [11].
- On the other hand, while computing in an approximate way the values of BE corresponding to the obtained values of $\langle r \rangle$, we found some discrepancy with the values deduced from the algebraic expression E_D. In fact, these values, as in the tables of [12], are obtained with an equation in pure Coulomb potential. To correct the discrepancy, we used a method of iterative computation with convergence, which leads to the following results [5]: for $k = 1, 2, 3, 10$, we respectively obtain BE = - 56 keV, - 275 keV, -301 keV, -320 keV, corresponding respectively to $\langle r \rangle = 12, 1.65, 1.39$, and 1.22 F.

We can observe two things:

- There is an important gap between the result associated with $k = 1$, and those associated with values of $k > 1$.
- The value obtained for $k = 1$, i.e. BE $= -56$ keV and $\langle r \rangle = 12$ F, seem to be similar to a result obtained by [13] with a very different method, indicated in Section 5.2.1

3. What is the Sign of the EDOs Solutions? We Show below it is Positive

As a consequence of our previous publications, it has been stated that EDOs represent in fact negative energy states. We have looked for and present, in this section, a clear solution to this issue.

First, we can note that the final steps to compute the energy levels of the solutions of relativistic equations lead to a relation defining the square of the energy E, i.e. $E^2 = \langle \text{expression} \rangle$. In these conditions, one has two alternatives for the expression of E, either to take the positive square root of $\langle \text{expression} \rangle$ or to take its negative square root. But most of time, in the textbooks, the positive root is systematically taken without discussion, since one implicitly supposes that the energy of regular energy solutions cannot be negative.

What about "singular solutions" such as EDOs? One knows the solutions of Dirac equation applied to electrons in an atom form two continuums: the electrons (positive energy continuum) and the positrons (negative energy continuum). It is the cause of the well-known Brown–Ravenhall disease [14] (also referred as the "continuum dissolution" problem), which happens while applying the Dirac equation for at least two atomic electrons. On double excitation of a pair of correlated electrons, one electron can end in the negative energy continuum (positrons), while the other lands in the positive energy continuum, the total system energy being retained. As a consequence, the number of such electron-positron states for an initial electron pair is infinite, and any energy level of the bound electron pair system is infinitely degenerate. Of course, as noted by Rusakova [15], this problem appears only in many-body cases. Nevertheless, it is quite legitimate to ask the question of the sign of the EDO energies, and as we indicated above, this sign cannot be deduced from the relation $E^2 = \langle \text{expression} \rangle$. In fact, if we look more attentively at the solution process of Dirac equation, we can observe the energy E, as the variable in an eigenvalue problem, is included in some parameters of the radial equations, after putting it in a pure numerical form. Most interesting: E appears in these parameters without square. For example, in [16], the author ends up to the following relation:

$$\frac{E\gamma}{\hbar c} = \alpha(n' + s), \qquad (3)$$

where γ is the coupling constant (usually noted α), while α is here a positive numerical constant, s the usual parameter defined by $s = \pm(k^2 - \gamma^2)^{1/2}$, taken here with the sign "+" (for regular solutions), and n' the radial quantum number. With s always positive, as $\gamma^2 \ll k$, the author says " ... $\langle$the above relation$\rangle$ incidentally shows that $E > 0$". But, in fact, one can more generally say: if the sign of s is not fixed, E has the sign of "$n' + s$".

Nevertheless, one can find another method to solve the Dirac equation, where one obtains the same result without needing to look for any auxiliary numerical parameters in the solution process. We refer here to the well-known method used in [10] and based on the transformation of the system of first order differential radial equations into a second order differential Kummer's equation. The solutions are in form of confluent hyper-geometrical functions. The final step to obtain the expression of the energy E, is carried out by solving the following algebraic relation:

$$\frac{1}{2}\alpha \left\{ \sqrt{\frac{mc^2 + E}{mc^2 - E}} - \sqrt{\frac{mc^2 - E}{mc^2 + E}} \right\} = (n' + s) , \qquad (4)$$

where n' and s have the same meaning as above, but α is the coupling constant. It is easy to see that the left side has the same sign as E, and thus E has the sign of $(n' + s)$.

Now, we return to the sign of EDOs energy solutions. The EDOs are obtained when $|k| = n'$ and $s < 0$, i.e. $s = -(k^2 - \alpha^2)^{1/2}$. As $\alpha \ll 1$, one has $n' + s = |k| - (k^2 - \alpha^2)^{1/2}$ and $|k|$ is always greater than $(k^2 - \alpha^2)^{1/2}$, thus $n' + s > 0$, we can affirm *the energy E is positive*.

Furthermore, we have, as already mentioned in our previous papers:

  – If $|k| > n'$, the energy E is < 0, so it does not concern physical solutions.

– If $|k| < n'$, the energy E is > 0. We have named "pseudo-regular" [1] the corresponding solutions, because they have energy levels near the regular atomic solutions. We can conclude that the case $|k| = n'$ separates the positive- and negative-energy solutions and it falls on the positive-energy side.

4. Role of Special Relativity for the Existence of EDOs

4.1. Special Relativity generates EDOs

In the recent papers [3,6], we reveal and analyze the essential role of Special Relativity for the possibility of EDOs among "singular" (i.e. not regular) solutions of quantum equations.

From the beginning of our study of these singular solutions, including the works of M&V [9,12], we had observed a pure algebraic argument, which allows us to sort the EDOs in the set of singular solutions. As recalled in Section 3, the energy levels E of the singular solutions of Dirac equation, are given by the algebraic expression (Eq. (2)), again written here:

$$E = mc^2 \left[1 + \frac{\alpha^2}{\left(n' - \sqrt{(k^2 - \alpha^2)} \right)^2} \right]^{-1/2}. \tag{5}$$

In the case where $|k| = n'$, one can easily show the expression of E is drastically reduced into $E \sim mc^2\alpha/2n'$. This leads to a binding energy $\mathrm{BE} = E - mc^2 \sim -mc^2(1 - \alpha/2k)$, whose absolute value is close to the rest mass energy mc^2. This is the signature of deep orbits. One can observe a similar result in the case of the relativistic Schrödinger equation: the EDOs are obtained when the quantum angular l and radial n' are equal. But, to find a deeper reason behind this pure algebraic fact, we have compared the two versions, relativistic and non-relativistic of the Schrödinger equation, as extracted from [16]. Doing this, we observed they have the same form, and only differ by the occurrence of a term including the coupling constant α. It is sufficient here to give the expression of the relativistic version, under dimensionless form, which reads:

$$\frac{1}{\rho^2} \frac{\mathrm{d}}{\mathrm{d}\rho} \left(\rho^2 \frac{\mathrm{d}R}{\mathrm{d}\rho} \right) + \left[\frac{\lambda}{\rho} - \frac{1}{4} - \frac{l(l+1) - \alpha^2}{\rho^2} \right] R = 0. \tag{6}$$

In fact, the non-relativistic version does not include the term α^2/ρ^2. Of course, the numerical parameter noted λ and occurring in this equation, has not the same expression as the similar one occurring in the non-relativistic version.

So we have "tracked" the term α^2/ρ^2 including the very small "salt grain" $\alpha^2 \sim 5.3 \times 10^{-5}$ brought by Relativity into the Schrödinger equation, which makes all the difference. Then we finally found it comes from the following "energy" factor, used while building the relativistic equation:

$$(E - V)^2 - (mc^2)^2 = E^2 - m^2c^4 - 2VE + V^2. \tag{7}$$

The left-hand-side expression is itself an intermediate expression coming from the relativistic expression of the total energy, after replacing E by $(E - V)$ to introduce the action of the "exterior" Coulomb field on the electron, into the equation in covariant form with minimal coupling. More precisely, the term α^2/ρ^2 originates from the far-right term above, V^2: the square of the Coulomb potential energy. On the other hand, we can observe that α^2 also occurs in the algebraic expression of the level energies for singular solutions, both for the (relativistic) Schrödinger equation and for the Dirac equation, and it is the source of the EDOs (by a dramatic simplification, as explained above). Moreover, we note the two following things:

(1) Expression (7) contains in a relativistic way, i.e. in quadratic form, the energy of the electron in a central Coulomb field minus the rest mass energy. So, the insertion of the energy term is under quadratic form $(E - V)^2$, instead of a linear form $(E - V)$ for the non-relativistic case. We can observe a kind of "entanglement" of E and V due to the non-linear form of expression of the relativistic (total) energy.

(2) The term V^2 corresponds in fact to a dynamic relativistic correction of the Coulomb potential V. This correction (see e.g. [17,18]), which is also provided by the Dirac equation (in a less visible way), leads to an effective "dynamic" electric potential given by the general following formula:

$$V_{\text{eff}} = V(E/mc^2) - V^2/2mc^2. \tag{8}$$

If considering relativistic electrons, this expression can be rewritten under the following form:

$$V_{\text{eff}} = \gamma V + V^2/2mc^2. \tag{9}$$

Here γ denotes the well-known relativistic coefficient, i.e. $\gamma = (1 - \nu^2/c^2)^{-1/2}$.

It is not obvious, from both expressions of V_{eff}, to see one has always a strengthening of the normal Coulomb potential, i.e. $|V_{\text{eff}}| > |V|$. Nevertheless we have shown in [3] that it is true in the case of quasi-circular orbits, more precisely:

   – $|V_{\text{eff}}| > |V|$ and V_{eff} is attractive: i.e. $V_{\text{eff}} < 0$.
   – $|V_{\text{eff}}|$ quickly increases as a function of $|V|$, with a parabolic behavior in $|V|^2$ when $|V| \to +\infty$.

Moreover, while computing V_{eff} from results [5] concerning EDOs in atomic H, approximated by considering almost circular orbits for comparison purposes, we can see the following:

   – When k increases, and as the mean radius of the orbits decreases, then γ increases and $|V_{\text{eff}}|$ increases too.
   – The ratio $|V_{\text{eff}} - V|/\text{KE}$ increases with the kinetic energy KE. So, it seems we have a kind of entanglement between mass, kinetic energy, and potential energy, as Special Relativity mixes total energy and potential energy in a non-linear way: it is a similar observation as we noted above, about the effect of the non-linear energy term $(E - V)^2$.

Here we illustrate this result, by giving only two examples about EDOs extracted from [6]. We indicate approximated values of $\langle r \rangle$, V, γ, V_{eff}, the corresponding potential energy shift $|\Delta V| = |V_{\text{eff}} - V|$ and the kinetic energy KE:

   • $k = 1 \Rightarrow \langle r \rangle \sim 6.6$ F, $\gamma \sim 1.2$, $V \sim -218$ keV, $V_{\text{eff}} \sim -222$ keV, $|\Delta V| \sim 4$ keV, KE ~ 120 keV.
   • $k = 10 \Rightarrow \langle r \rangle \sim 1.2$ F, $\gamma \sim 2.7$, $V \sim -1.17$ MeV, $V_{\text{eff}} \sim -1.79$ MeV, $|\Delta V| \sim 620$ keV, KE ~ 854 keV.

4.2. What about the singular solutions in the non-relativistic case?

Now, after having recalled how one can see how Special Relativity induces the existence of EDOs, we examine the case of the singular solutions in the non-relativistic case, essentially while considering the (non-relativistic) Schrödinger equation. There exist different (and equivalent) forms of this equation. Here, we consider the form used in [16]: the radial equation (16.7) having, as indicated above, the same form as (6), but without the term α^2/ρ^2. The parameter noted λ', instead of λ in (6) to avoid confusion, is defined by the following expression:

$$\lambda' = (e^2/\hbar)(m/2|E|)^{1/2}. \tag{10}$$

The ansatz used to solve the Schrödinger equation involves, as usual, a factor of the form $F(\rho) = \rho^s L(\rho)$, where $L(\rho)$ is a series in powers of ρ. It leads to a condition on the parameter s for convergence of the solutions when $r \to +\infty$, which reads:

$$s(s + 1) - l(l + 1) = 0. \tag{11}$$

This equation in *s* has two solutions:

  – If taking the root $s = l$, one has the regular well-known solutions.
  – Whereas, if taking $s = -(l + 1)$, one obtains singular solutions.

In the non-relativistic context, the total energy does not include the rest mass energy and is negative; moreover it represents the binding energy too. The values of energy values are expressed by means the following relation:

$$|E| = mc^2 \alpha^2 / 2\lambda'^2. \tag{12}$$

As showed in [6], it is not possible to put $l = n'$ to obtain EDO's, because this would give a null denominator. But this only shows one cannot apply, to the non-relativistic equation, the recipe used to find EDOs with the relativistic one. Here we can indicate, in a simple way, the necessity of the relativity to obtain EDOs. Based on the solution method used in [16] and even if the recurrence relations for the case $s = -(l + 1)$ are different from the case $s = l$, it clearly appears that the series $L(\rho)$ must terminate to provide the convergence of the solutions for $\rho \to +\infty$. This implies λ' has to be an integer, as in the regular case. Under these conditions, we can see the maximum (absolute) value of the binding energy is obtained for the minimal possible value of the integer $|\lambda'|$. If this minimal value is 1, we have in fact $|E| = mc^2 \alpha^2 / 2$, where we recognize the expression of the Rydberg energy ~13 eV, i.e. the ground state energy associated with the Bohr radius of 53 pm.

In conclusion, we have the following result: *with the classical Schrödinger equation for atomic H, one has no EDO.*

Putting together the positive result deduced from the relativistic case with the negative result deduced from the non-relativistic case, we can "surmise" the following global result: Special relativity is sufficient to obtain EDOs, and reciprocally, the existence of EDO needs Special relativity, i.e. Special relativity is necessary for EDOs. That can be written in a more symbolic way: Special Relativity $\Leftrightarrow$ EDO. Of course, as we suppose the electron is in the Coulomb electric potential of a nucleus, we wrote only "surmise".

4.3. Discussion

In the literature, one can find some results about singular solutions with a non-relativistic Schrödinger equation, but it seems they require a Coulomb potential modified by an *external confining action*, as e.g. in [19], to really obtain deep orbits. Indeed, in that article, the author considered a "compressed" atom in a confinement potential described by a finite potential step at some given radius. In a further article [20], the same author obtains a continuum spectrum of singular bound states, by solving the radial Schrödinger equation in Coulomb field in integral form (integral on closed paths in the complex plan), by using the Laplace method. Those singular wave-functions could contribute to screen the nuclear charge in a zone of radius $10^{-4} \times$ Bohr radius, i.e. ~5.3 F. That paper does not give the mean radius of the electron orbits, nor of their binding energy, but yields probabilities of population of the deep states leading to a compact "pico-atom", depending on the total energy E. That probability seems to go from $\sim 2 \times 10^{-6}$ to 6.6×10^{-5}.

It is very difficult to compare those results with the EDOs obtained by the relativistic equations. Nevertheless, that result can be related to the one of [21] for a computer simulation of hydrogen atom behavior in a flow of free electrons: during this simulation, the formation of "mini-atoms" is observed, with a smallest size of ~10 F. The incoming free

electrons have a kinetic energy of $\sim$3.6 eV, which is the average energy of conduction electrons in palladium. If confirmed, this study could help to suggest a way for reaching EDO states, while taking into account strong magnetic interactions (see Section 5) near the nucleus.

5. Survey on the Magnetic Interactions

At atomic levels, the magnetic interactions such as spin–orbit interactions or spin-spin interaction, have little effect on the energy levels of bound electrons, leading only to very small energy shifts. Their essential interest is to break the degeneracy of energy levels and split the spectral lines. Nevertheless, at deeper levels, they can become very strong and even be dominant over the Coulomb interaction. Below, without claiming to be exhaustive, we recall some elements about the magnetic interactions; then we analyze several works on the magnetic interactions, the methods to account for their actions at short distances, and the difficulties encountered in these questions.

5.1. Elements on magnetic interactions

We recall some simple and necessary elements of knowledge about magnetic interactions in an atom. We only consider the atomic H, or D (when we explicitly indicate it).

5.1.1. *Spin–orbit interaction*

When taking into account the spin of a bound electron, one has the well-known Spin–Orbit (SO) coupling: it has the greatest observable effect, since it is involved in the "fine-structure" effect, a splitting of the spectral lines, and it partially breaks the degeneracy of the atomic energy levels. The Spin-Orbit interaction is the cause of only a part of the energy correction leading to the fine-structure effect, as further corrective terms are involved: a kinetic energy relativistic correction at first-order in v^2/c^2 and the Darwin term. In fact, these three corrective terms are automatically included (but hidden) in the Dirac equation. We do not expose specifics showing this result, as it is in most QM textbook and courses. We recall that, in fact, one considers the relative orbital movement of the nucleus in the electron reference frame, which gives a magnetic field "seen" by the electron. So, on account of the magnetic moment associated with the electron spin, this relative nuclear magnetic field causes a spin precession (Larmor), expressed in a quasi-classical way by a potential energy. But one has also to take into account the further Thomas precession, having a pure relativistic nature and acting in the opposite direction of the precession. If the electron is only slightly relativistic, the energy of Thomas precession is half the Larmor precession.

The general quantum expression of the spin-orbit interaction reads:

$$H_{\text{SO}} = \frac{1}{2\,m^2 c^2} \left(\frac{1}{r} \frac{\partial V}{\partial r} L \cdot S \right), \tag{13}$$

where L and S respectively denote the electron angular momentum and spin operators.

As V is the Coulomb potential energy, by replacing V by its expression as a function of r, one obtains:

$$H_{\text{SO}} = \frac{\mu_0 e^2}{8\pi m^2} \frac{1}{r^3} L \cdot S = \xi(r) L \cdot S. \tag{14}$$

Moreover, the scalar operator $L \cdot S$ commutes with the total angular moment $J = L + S$ and the potential expressed by (14) commutes with L^2, S^2, J^2 and J_z. By using known rules on composition of angular moments and Clebsch–Gordan coefficients, one shows $L \cdot S$ can be expressed by means of the eigenvalues j, l, and s (fixed $= 1/2$). If $l \neq 0$, one can write:

$$L \cdot S = \frac{\hbar^2}{2} \left[j \left(j + 1 \right) - l \left(l + 1 \right) - s \left(s + 1 \right) \right] = \frac{\hbar^2}{2} \left[j \left(j + 1 \right) - l \left(l + 1 \right) - 3/4 \right]. \tag{15}$$

Here $j = l \pm 1/2$ and

- – If $l = 0$, there is no SO interaction.
- – For any value of $l > 0$, one has two cases:
- • $j = l - 1/2 \Rightarrow L \cdot S = -(\hbar^2/2)(l+1)$, that gives an attractive potential, since $\xi(r)$ is > 0.
- • $j = l + 1/2 \Rightarrow L \cdot S = +(\hbar^2/2)l$, giving a repulsive potential.

To obtain numerical values for the SO potential energy as function of $\langle r \rangle$, one replaces the physical constants occurring in $\xi(r)$ by their values in a chosen unit system, e.g. the SI standard. Then the average energy $\langle E_{\text{SO}} \rangle$ for $l = 1$ reads:

$$\langle E_{\text{SO}} \rangle \sim (1.71 \times 10^{-53}/\langle r^3 \rangle)\ \text{J} \sim (1.07 \times 10^{-34}/\langle r^3 \rangle)\ \text{eV}.$$

As an example, we give a quick computation: for the 2p orbital (of atomic H), where $\langle r \rangle \sim 10^{-10}$ m, $\langle E_{\text{SO}} \rangle \sim 1.06 \times 10^{-4}$ eV. But we can easily see that, for a very small radius, the energy becomes very high. For example, with $\langle r \rangle \sim 10^{-13}$ m = 100 F, one has $\langle E_{\text{SO}} \rangle \sim 106$ keV. Note this generalization is made, by supposing the rules for angular moments can be extended for any radius. Moreover, when high energies are concerned, we have to take into account relativistic corrections. Finally, we remark the following things:

- – The SO coupling is based on the relative motion of the proton with respect to the electron, but one considers the *angular momentum operator L of the electron* to compute it.
- – The composition of angular moments, not totally developed here, leads to a computation of the mean value of the energy by using the eigenvalues while avoiding the integration of the operator with a wavefunction.

5.1.2. *Magnetic interactions involving the nuclear spin*

In taking into account the nuclear spin, we have further magnetic interactions, which are involved in the so-called "hyperfine structure" of the atomic spectral lines and further breaking of the degeneracy of the energy levels. The analysis of these magnetic interactions is complex, but we only need deal with the energy. Therefore, we can make some simplifications while evaluating orders of magnitude of the coupling energies. For example, there is a spin–orbit interaction $S_{\text{p}}O$ concerning the nuclear spin and the (direct) orbital motion of the electron represented by L. It can be repulsive or attractive, depending on the relative directions of the nuclear spin and of the electron angular momentum. As the magnetic moment of the proton is $\mu_{\text{p}} \sim 1.41 \times 10^{-26}$ SI units, whereas for the electron it is $\mu_{\text{e}} \sim 9.3 \times 10^{-24}$ SI units, we have a ratio $\mu_{\text{e}}/\mu_{\text{p}} \sim 670$. Under these conditions, for any combination of type (repulsive/attractive) of interactions SO and $S_{\text{p}}O$, the coupling energy of $S_{\text{p}}O$ can be neglected relative to SO.

For the same reason, another magnetic interaction with the nuclear spin, the spin–spin interaction, which is much weaker than the SO interaction, will be neglected when we take into account SO. Nevertheless, if $L = 0$ one has neither SO nor $S_{\text{p}}O$ coupling. Thus, one has to take into account the interactions between the nuclear and electron spins. We have the following expression, including the respective magnetic moments M_{p} and M_{e} of the proton and the electron, due to their spins:

$$H_{\text{SS}} = -\frac{\mu_0}{4\pi} \left[\frac{1}{r^3} \left(3 \left(M_{\text{p}} \cdot \hat{r} \right) \left(M_{\text{e}} \cdot \hat{r} \right) - M_{\text{p}} \cdot M_{\text{e}} \right) + \frac{8\pi}{3} M_{\text{p}} \cdot M_{\text{e}} \delta \left(r \right) \right], \tag{16}$$

where $\hat{r}$ denotes a radial unit vector.

As the spins are involved in the magnetic moments, one has to consider a combination of both electron and proton spin operators S_{e} and S_{p}. By introducing the total spin $S = S_{\text{e}} + S_{\text{p}}$, one can write the following relation:

$$S_{\text{e}} \cdot S_{\text{p}} = 1/2(S^2 - (S_{\text{e}})^2 - (S_{\text{p}})^2) = (\hbar^2/2) \left[s(s+1) - 3/2 \right], \tag{17}$$

where the only possible values of s are $s = 1$ ("triplet" state) and $s = 0$ ("singlet" state). So, we have two cases:

- $s = 0 \Rightarrow S_e \cdot S_p = -(3/4)\hbar^2$.
- $s = 1 \Rightarrow S_e \cdot S_p = +(1/4)\hbar^2$.

Here, we wrote only a little part of the computation (which is in most QM books/courses) and we note that the case $s = 0$ gives an *attractive* interaction whereas $s = 1$ a *repulsive* one. In fact one can use the following relation for the energy E_{SS} of the spin–spin interaction in the attractive case: $\langle E_{SS} \rangle = -(3/4)A(r)$ where, for $\langle r \rangle = a_0$ (the Bohr radius), $A(a_0)$ is proportional to $1/a_0^3$ and has a value $\sim 5.87 \times 10^{-6}$ eV for the fundamental level 1s of the hydrogen atom. In these conditions, to make approximate computations (Section 6), we extrapolate this result to a general expression of the spin-spin interaction energy E_{SS} for any radius $\langle r \rangle$:

$$\langle E_{SS} \rangle \sim (1 \times 10^{-55}/\langle r^3 \rangle)\,\mathrm{J} \sim (0.64 \times 10^{-36}/\langle r^3 \rangle)\,\mathrm{eV}. \tag{18}$$

As for the SO coupling and when high energies are concerned, we have to take into account relativistic corrections. At high speed considered in the observer reference frame, the direction of the spin appears to realign itself closer to that of the translational velocity vector [22] which is normal to the orbital angular momentum and is time varying, with an average value approaching zero as $v \Rightarrow c$: this should imply a weakening of both the $L \cdot S$ of the SO term and the $S_e \cdot S_p$ and $1/\langle r^3 \rangle$ dependence of the energy E_{SS}.

5.1.3. *Case of a term in 1/r^4*

In the next section, we identify a term including the square of the magnetic vector potential, i.e. $\mathbf{A}^2$, generally neglected at atomic levels. It occurs, e.g., while developing the hyperfine structure Hamiltonian or a Pauli equation. In both cases, it comes from a term of the form $(\mathbf{p} \pm q\,\mathbf{A})^2/m$. As it includes the vector potential, it is a term of magnetic interaction: it expresses a diamagnetic energy. We give further the expression used to compute this energy term.

5.2. The Vigier–Barut model

In this section, we speak about works referring to the "Vigier–Barut model", but we first specify the origin of this naming.

5.2.1. *Works of Barut, with a "more complete" Dirac equation*

The Vigier–Barut model and the works related to this model, derive from works of Barut, e.g. [23,24]. In these articles, the author looks for an analytic solution of the Dirac equation for a charged lepton with anomalous magnetic momentum (AMM) in Coulomb potential. Note that including the AMM into the Dirac equation, which contains already the "normal" magnetic momentum of electron, is in fact a way to take into account a second order QED contribution: more precisely a one-loop correction to the fermion magnetic moment. For doing this, the author uses a Dirac equation, where the vector potential is completed by a term introducing the electron AMM and expressed by means of the EM tensor $F\mu\nu$ and Dirac matrices γ. As pointed out by Barut, the AMM of the electron is small relative to the normal magnetic moment, but it is of the same order-of-magnitude as the magnetic moment of the proton. After numerous and complex transformations, he obtains a second order differential radial equation. This equation has a form similar to a radial Schrödinger equation, with a specific dynamic potential V expressed as sum of inverse powers of the radius r, plus a term k^2 depending on E^2, where E represents the energy eigenvalue. More precisely, the differential equation reads:
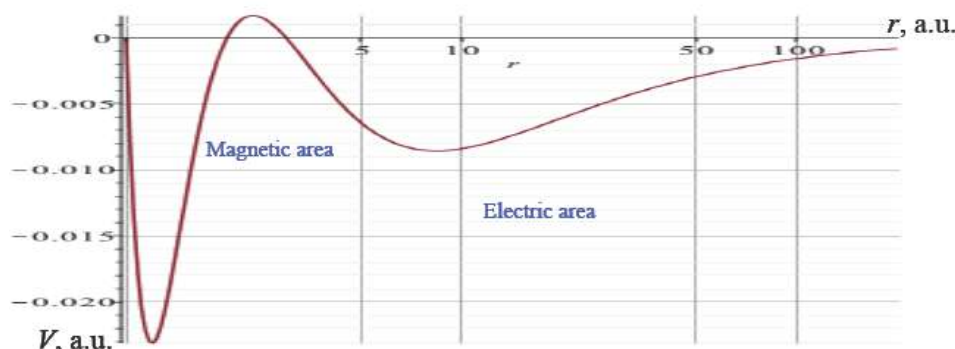
Figure 1. Potential with two wells.

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - V(r) + k^2\right)\psi = 0. \tag{19}$$

The potential V is expressed by

$$V(r) = A/r + B/r^2 + C/r^3 + D/r^4. \tag{20}$$

Here *A, B, C* and *D* are built from physical constants, the mass or the reduced mass, E, and initial parameters of the equation, written in natural units, i.e. with $\hbar = c = 1$. In the second cited paper, these coefficients are constant for any given eigenvalue E. We have to note the occurrence of E in a non-linear form that is the "signature" of Relativity

Note that in the "dynamical" potential $V(r)$ computed by Barut, the expressions of the coefficients B, C and D are complex, energy-dependent, and difficult to interpret physically.

But, on the other hand, in the works associated with the so-called Vigier–Barut model:

– The term in $1/r^2$ has clear meaning of the well-known "centrifugal barrier" energy.
– The term in $1/r^3$ represents magnetic interactions, such as those described in Section 5.1:

  – either spin–orbit, available only if the angular quantum l is not null. It is included in the Dirac equation. It can be attractive or repulsive, depending on the total angular number j
  – either spin–spin, involving the nuclear spin, which is not included in the Dirac equation but can be involved in the Pauli equation. It can be attractive (case of "singlet" state) or repulsive (case of "triplet" state).

– The term in $1/r^4$ is the least known. It has meaning as a "diamagnetic" term and involves the vector potential **A**. It is always repulsive and can be deduced from the Pauli equation. Some explanations are given in Section 5.1.3 and especially in the body of Section 5.2.2.

In fact, Barut does not apply Eq. (19) to the atomic H, but to a coupled system (e+, e–) called "positronium." Doing this, he finds two potential wells as pictured in Fig. 1, and he looks for resonance states of positive energy. Note V reaches a maximum at the huge energy $V_{\max}$ of order 35 GeV, when the radius becomes very small and orders-of-magnitude smaller than the region corresponding to the regular (atomic) positronium bound state. Then Barut raises the abscissa axis to $V_{\max}$ to treat the decay of the resonances. In conclusion, he summarized the following result: there

are new resonant states at distances of order αr_0 of very large positive mass–energy of order GeV; here, r_0 denotes the classical electron radius ~ 2.8 F, which gives $\alpha r_0 \sim 0.02$ F. Moreover, he says the used model could describe the electron–proton system and give (very) deep bound electrons, while noting the proton (finite) size leads to a further effect. However, it is difficult to imagine a deep electron resonance being at a distance 0.02 F from the origin, since it would be inside the proton.

In another paper [25], the author uses a very different method to treat tight molecules and finds an inter-atomic binding energy of ~ 50 keV. On the other hand, in [26] he extends his model, by taking into account the short-ranged strong magnetic forces, to the high-energy physics, including strong and weak interactions.

Note an article of Behncke [27], where the author analyzes the properties of the Dirac equation for an electron in a spherically symmetric electric potential V, under a general linear covariant form including the electron AMM. While proving the Hamiltonian H is essentially self-adjoint, it appears the AMM term induces a repulsive force, regardless of the sign of V, which forces the wavefunctions in the domain of H, to vanish at the origin. This latter result is indicated in [28], under the form of a further term in V' in the usual radial Dirac operator.

5.2.2. *Works associated with the Vigier–Barut model*

Regardless of the relativistic origin of the model, we speak now about some works carried out in a non-relativistic framework: we can cite the research of Vigier [29], Samsonenko et al. [13], Dragic et al. [30], on possible tight orbits under the Bohr level for atomic H (or D), and even for tight molecules. Of course, this list does not exhaust the concerned subject. All these works are attempts to explain LENR results by screening effects in "tight" orbits, in particular the initial experiment of Pons and Fleishmann [31], which was recent at the time. We set aside the work of Ozcelik et al. [32], which does not aim at finding tight orbits, but is referred to by the previous cited works, because it presents an exact solution of the radial Schrödinger equation for inverse-power potentials. But, as its purpose is different and concerns mainly the interatomic interactions, the solution method contains some restrictions on the coefficients of the potentials, which limit the possibilities of application for the intra-atomic magnetic interactions. All these works have in common the use of a potential V defined as a sum of inverse powers, similar to the expression (20), with constant coefficients A, B, C and D.

For the first three cited papers, the starting point is a two-body Hamiltonian for the electron and the nucleus, taking into account the magnetic moments of the two particles and including, as usual, the momentum **P**. This Hamiltonian is inserted into a Schrödinger-like equation. So interactions such as spin–spin and spin–orbits can be explicitly expressed. In fact, the obtained equation is an extension of the (non-relativistic) Pauli equation to a two-body system of charged particles. Next, algebraic transformations and developments applied to the equation lead to a new equation including a potential V expressed as in (20). Note that, while defining the reduced mass, the authors of [30] explicitly talk about a proton, which means they only handle the atomic H. In [29], the author explicitly speaks about hydrogen and deuterium, and in [13] the authors speak about particles without specifying them, but as they refer to the work of [29], one can suppose their study can be applied to H or D.

We can see, in the potential V, the term in $1/r$ is the Coulomb potential and the term in $1/r^2$ is the classical term in $l(l + 1)$ expressing the centrifugal barrier. The term in $1/r^3$, characteristic of the magnetic interactions spin–spin and spin–orbit, are similar in the different works. The term in $1/r^4$ could seem weird, but by coming back to the initial Hamiltonian, we can see it comes from terms equivalent to $(\mathbf{P}_i - q_i \mathbf{A}_j)^2/m_i$; $(\mathbf{P}_i - q_i \mathbf{A}_j)^2$ is the square of what Feynman names "dynamical momentum" or "p-momentum" [33], including the coupling of one particle with the EM field of the other particle. The expression of the vector potential produced by a magnetic dipole moment **m** reads:

$$\mathbf{A}(r) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{r}}{|\mathbf{r}|^2}.$$

(21)

Here $\hat{r}$ is the unit vector in the direction of the radial vector $\mathbf{r}$ and μ_0 is the vacuum permeability. So, the square of the p-momentum generates a term in $1/r^4$. Note that, to take into account the magnetic interactions in a symmetric way, the Hamiltonian must include the p-moment of each particle (e.g. electron and proton) and the dipole–dipole interaction terms.

One can ask what is the physical meaning of the square $\mathbf{A}^2$, included in the term D/r^4. But the complete term $q'^2\mathbf{A}^2/m$, having the dimension of an energy, is considered [34] as expressing a diamagnetic energy. But it would be interesting to deepen its physical meaning. Moreover, we can also cite an article [35] about the significance of $\mathbf{A}^2$ in QFT context, but it exceeds the framework of our paper.

Here, it is not superfluous to say some words about the vector potential: It allows us to greatly simplify the writing of relations and equations of Electromagnetism; the electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields can be directly derived from $\mathbf{A}$, the Coulomb gauge is simply $\nabla \cdot \mathbf{A} = 0$ and the Maxwell's equation in this gauge is $\partial_\mu \partial^\mu \mathbf{A} = 0$, the Lorenz gauge can be written $\partial_\mu A^\mu (x) = 0$ and so the Maxwell's equation in Lorenz gauge can be written $\partial_a \partial^a A^\mu = 0$. On the other hand, the concept of vector potential has a proper scientific interest, whereas in the past, famous physicists supposed it was only an abstract mathematical tool without physical reality. Nevertheless, it became necessary to explain some observations in the EM theory, e.g. the well-known and strange phenomenon called Aharonov-Bohm effect [36], which shows the local fields $\mathbf{E}$ and $\mathbf{B}$ do not contain full information on the EM field.

To return to the term D/r^4: this term is generally neglected, as having a very small value at the atomic levels. But D/r^4 has an important role in the Vigier–Barut model: as D is > 0, D/r^4 acts as a repulsive core at small distances of the nucleus, while the Coulomb potential A/r is attractive at large distances. B is always > 0, since B/r^2 represents the centrifugal barrier, and C/r^3, including magnetic interactions, can be > 0 or < 0. With these conditions, if $C < 0$, we have the following succession of potentials, intentionally written in the order of the decreasing powers of r:

$$D/r^4 \text{ [repulsive]}, \quad C/r^3 \text{ [attractive]}, \quad B/r^2 \text{ [repulsive]}, \quad A/r \text{ [attractive]}.$$

For some combinations of values of the coefficients, if $V(r)$ has three zeros, i.e. there are three different values r_1, r_2 and r_3 such that $V(r_1) = V(r_2) = V(r_3) = 0$, *then $V(r)$ includes two wells*:

- The first well, for the small values of the radius r, corresponds approximately to a particular zone where the magnetic interactions are dominant: one can call it the "magnetic region" as in [37]
- The second well, occurring when r increases toward the Bohr radius, corresponds to a zone where the electric Coulomb potential is dominant: the "electric region" [37]

In Fig. 1, we represent a pure abstract mathematical exercise to show, in a simple way, that some combinations of coefficients can give rise to two local minima for the plotted curve of a potential like V. This curve does not correspond to the computation of an actual physical case. Here, as well as in [29] and in these conditions, the values on the axes have no physical units. Plotting actual values, with even a limited range from 1 eV to ± 1 MeV against 1 nm to 1 F, is a bit awkward. It would have to be done on a modified log-log plot ($> 7 \times 6$ cycles). Therefore, we only illustrate such a situation in Fig. 1 by plotting, in semi-log, a simple simulation of a representative potential V satisfying the above conditions of Eq. (20).

In fact, the methods used in the cited works are slightly different, so some terms can be neglected because of the complexity of the computations and the different approximation methods (e.g. the "hard core" approximation is used in [30]). The analytic methods meet numerous difficulties, which lead to solving the equation numerically. With these circumstances, as written in [13]: "there is a set of contradictory estimates of energy values and quantum-orbit radii based on the starting Hamiltonian." It is not a criticism of these works, because they are important and difficult studies in a field that has been barely explored, if at all: the domain of the magnetic interactions near to the nucleus.

We note Vigier [29] also develops a theoretical study for computation of new tight orbits, based on the causal mathematical formalism of L. deBroglie [38] and Bohm [39]. Unfortunately he does not give numerical results on

these tight orbits. He indicates that he would continue this study in a future article, which seems to have never been published. On the other hand, his work [29] includes a long study about a three-body problem, by considering possible new "tight" molecules, i.e. one noted H_2^+ and another D_2^+, constituted from two nuclei aligned with an electron between them, while the nuclei rotate around the electron. It can be compared to a similar work of Barut [25], where the author also considers linear molecules, but in a triangular structure, where the electron oscillates in the plane perpendicular to the line joining the nuclei. For these tight molecules, Vigier computes *numerical results* for energies of ground states and he obtains ∼28.1 keV for H_2^+ and ∼56.2 keV for D_2^+. Barut indicates similar size order of energies, e.g. ~50 keV for D_2^+.

To conclude with the numerical results, we can see

- In [13], the authors indicate a tight state of energy ∼40 keV. We recall that, as it is a study in non-relativistic context, this value is that of the total electron energy and the binding energy too.
- In [30], the authors give numerous results for the spectra of the hydrogen atom, at different energy levels and for different values of quantum numbers L, S, and J. Nevertheless, the indicated energies are similar to classical atomic values, i.e. a few eV. In fact, they did not find tight orbits with high energies of order keV.

In conclusion, we can note the results concerning tight orbits in these works seem to correspond to a result we found [5] for the "ground" level of EDOs, i.e. for $k = 1$, when applying a special process for coherence between the energy and the value of the mean radius, as indicated in Section 2.2: $|BE|$ ∼56 keV for $\langle r \rangle$ ∼12 F.

6. Physical Questions About EDOs, and First Attempts of Solution by Computations

Two important questions have not yet been resolved in our previous papers:

(1) Can the EDOs satisfy the HUR, a pillar of QM?
(2) Are the EDOs stable solutions? That is to say, do they have a lifetime sufficient to induce a fusion or transmutation process?

6.1. The question of the *HUR*. Relativistic confinement energies

Consider an electron confined near the nucleus, in a region delimited by a mean radius $\langle r \rangle$ of order ∼2 F, the size order of EDOs for atomic H. We compute an estimate of the kinetic energy of the electron: from HUR, $\Delta|\mathbf{p}|\Delta r \geq \hbar/2$. It is usual to approximate the uncertainty in $|\mathbf{p}|$ by $\Delta|\mathbf{p}| \sim \hbar/\langle r \rangle$ and to take for $\Delta|\mathbf{p}|$ an estimate of the momentum $|\mathbf{p}|$ of the confined particle. So we let $|\mathbf{p}| \sim \hbar/\langle r \rangle$, which gives $|\mathbf{p}| = 5.27 \times 10^{-20}$ SI units for the given $\langle r \rangle$. On the one hand, one sometimes finds a reasoning based on the classical kinetic energy for a particle in a Coulomb potential V, $E - V = \mathbf{p}^2/2m$; then one can deduce the value of $E - V \sim 1.5 \times 10^{-9}$ J ~ 9.5 GeV: one obtains an actual "hadronic" energy. On the other hand, as it makes no sense to talk of non-relativistic models in the nuclear region, consider the relativistic expression $E - V = [\mathbf{p}^2c^2 + m^2c^4]^{1/2}$. Then, with the previous value of $|\mathbf{p}|$, one has $|\mathbf{p}|c = 1.58 \times 10^{-11}$ J, while the electron rest mass $mc^2 \sim 8.2 \times 10^{-14}$ J. So, as $m^2c^4 \ll \mathbf{p}^2c^2$, one can write $E - V \sim |\mathbf{p}|c$ and KE $\sim E - V - mc^2 = 1.57 \times 10^{-11}$ J ~ 98 Mev, i.e. ∼100 times smaller than the value found by using a non-relativistic formula. The mathematical reason of this difference is well-known [40]: from the relativistic expression of $E - V$, we see the kinetic energy increase linearly with $|\mathbf{p}|$ for large momenta, whereas in non-relativistic mode it increases as $|\mathbf{p}|^2$

These are rather coarse computations, but the result is very interesting, because energy of about 100 Mev can be almost realized by the accepted spin–spin interaction alone (Section 5), which is still much weaker than the accepted spin–orbit interaction. Indeed, for $\langle r \rangle$ ∼2 F, we have $E_{SS} \sim 1 \times 10^{-55}/(2 \times 10^{-15})^3 \sim 1.3 \times 10^{-11}$ J ~ 81 MeV. With such values, one can think *the HUR is practically satisfied by the electrons on EDOs,* thanks to magnetic coupling. We

have yet to make more precise computations, by taking into account more parameters of interactions near the nucleus, as well as precise effects of relativity, which tends to bend the spin into the direction of the motion of the particle.

6.2. Question on the stability of the EDOs

An approximate but classical way to verify the stability of a particle state is to show it corresponds to a local minimum of total energy of the particle. For doing this, we start from the relativistic expression of energy E, and we replace the momentum $|\mathbf{p}|$ by $\hbar/r$, in order to include the HUR, and then we add a term V to E, to represent a potential energy. So, we have the following expression:

$$E_H = \sqrt{\frac{\hbar^2 c^2}{r^2} + m^2 c^4} + V. \tag{22}$$

Next we look for a sum of potential terms as in the Vigier–Barut model, to obtain a local minimum of E_H.

- In particular, we include in V a repulsive term in $1/r^4$, which, as indicated in Section 5, is essential to obtain a potential well near the nucleus with a repulsive hardcore. This term is computed by means of the following expression, deduced from the electron vector potential and including usual physical constants in SI units:

$$V_4 = K_4/r^4, \quad \text{where } K_4 = \left(\frac{\mu_0}{4\pi}\right)^2 \frac{\mathbf{e^4}\hbar^{\mathbf{2}}}{4 m_e^2 m_p} \sim 1.3 \times 10^{(-71)} \text{ SI}. \tag{23}$$

There is a symmetric term built with the proton vector potential, but it can be neglected in relation to the former one.

- One considers terms in $1/r^3$, for magnetic interactions, and the classical term in $1/r$ corresponding to the electric Coulomb potential energy $V_1 = -\alpha c\hbar/r$. So, at this stage, we have $V = V_1 + V_3 + V_4$, where V_3 is yet to be determined.

Initially, as the expression of E_H already includes a momentum-like term (under the square root), we do not include a centrifugal barrier term in V, because it already corresponds to a part of the Laplace operator in spherical coordinates, which introduces the dependence on the angular momentum l. Of course, by doing this, one partly eliminates the role of l.

On another hand, if we compute the potential energy $V_3 = E_{\text{SO}}$ associated with the spin–orbit interaction in attractive mode, while supposing $l = 1$ and applying the usual quantization rules, we obtain $E_{\text{SO}} \sim -13$ GeV for an electron at a distance $r \sim 2$ F, i.e. in the expected region of the EDOs. Such a huge value seems not physically reasonable. Moreover we obtain the deep local minimum of E_H at $r \sim 0.001$ F, i.e. totally inside the nucleus: it has no physical meaning since the used expressions for computing are no longer correct when reaching within the nuclear surface. Then, we tried to introduce the repulsive centrifugal barrier term in $1/r^2$ into V, in thinking it can have a restraining effect on the value of E_H near the nucleus. But we obtain again the deep local minimum at $r \sim 0.001$ F. We can also remark, for each value of n', the energy levels of EDO solutions of Dirac equation differ very little from those of the Schrödinger equation, although the Dirac equation takes the SO interaction into account: so one can think that E_{SO} does not act for the EDOs. Moreover, the SO coupling requires more hypotheses to be satisfied than the SS coupling: the spins are intrinsic parameters and the rules of spin addition are independent of the context. Thus, we do not take into account E_{SO}, but rather the potential energy E_{SS} corresponding to the spin–spin interaction in attractive mode. So, with $V_3 = E_{\text{SS}} \sim (1.04 \times 10^{-55}/\langle r^3 \rangle)$ SI, we put $V = V_1 + V_3 + V_4$ in E_H.

Here, we give some numerical results obtained with this potential:

- The deep local minimum of E_H is at $r \sim 0.17$ F, i.e. again inside the nucleus, albeit much less deep than with E_{S0}
- There is a local maximum of E_H of 42 MeV, at $r \sim 3$ F.
- At $r = 2$ F and assuming the orbit to be quasi-circular, we have $\gamma \sim 150$ and the kinetic energy KE ~ 76 MeV.

At this time, and after numerous further computation trials, we can observe that the repulsive diamagnetic potential V_4 is too weak to "push" the local minimum outside the nucleus. Moreover, if we computed an "optimal" value of K_4, to obtain a local minimum for $r \sim$ a few F, it would be of size order 10^{-68}, i.e. three orders greater than the present value. For example, with $K_4 \sim 3.4 \times 10^{-68}$ the deep local minimum is at $r \sim 2.6$ F. Of course, all these computations are made in a context where several further effects due to relativity and the nearness of the electron to the nucleus are not taken into account. For example:

- By simply considering the finite size of the nucleus and electron, one can think that the magnetic potentials, dominant in this region, overlap and decrease while approaching the nuclear "surface" defined by the charge radius, thus tending toward a finite limit.
- The tightly bound ultra-relativistic electron has its spin axis precessing about its velocity vector, which is normal to its orbital angular-momentum vector, so closely that the L·S and orbit-averaged S_n·S_e terms go toward zero.
- Moreover, QED effects such as electron self-energy and vacuum polarization are certainly strengthened in the extreme EM field near the nucleus. So one can expect a cloud of virtual lepton pairs (e+, e−) to surround the electron, causing a strong screening effect of the interactions.

With these effects, a potential V, highly attractive near the nucleus, could stop increasing (in absolute value) when approaching the nuclear surface and would decrease within: so one would have a potential well leading to a local minimum of the energy. Resonances between these effects would lead to the predicted quantized orbits and deep-energy levels.

Of course, such a qualitative reasoning requires confirmation by rigorous mathematical proof.

7. Conclusion, Open Questions and Future Work

Several questions are investigated here, in the continuation of our previous work.

- We quickly recalled our studies on the EDOs of the atomic H as solutions of relativistic quantum equations, in particular our review of the method of Coulomb potential corrected at the origin [9] which, we believe, counters all mathematical arguments against the EDOs and which we have extended with new results about general properties of EDOs.
- Then, from methods used to find deep orbits as singular solutions of relativistic equations, we have discussed a recent and subtle criticism of the sign of the energy levels of these solutions, which is of course a crucial question. So we showed, in a simple way, that the *EDOs have a positive energy E*. More precisely, we can see that the equality condition on angular and radial numbers, $|k| = n'$ (case of the Dirac equation) or $l = n'$ (Schrödinger equation), generates the EDOs as a particular class included in the set of singular solutions. This condition separates the positive energy solutions from the negative energy ones: if $|k| > n'$, $E < 0$, whereas for $|k| \le n'$, $E > 0$ including the case $|k| = n'$ which characterizes the EDOs.
- Next we gave new elements about the role of the Relativity for the deep orbits: from an analysis of singular solutions of the non-relativistic Schrödinger equation, we surmise Relativity is not only a source of the deep orbits, but is necessary to obtain them. Nevertheless, the question is not completely resolved if one considers atoms "compressed" by external potentials.

- But the heart of our paper is devoted to a study of the magnetic interactions near the nucleus, with the aim to solve two further important questions, of physical nature:
- Can the EDOs satisfy the HUR, which is a pillar of QM?
- Can we guarantee their stability?

As the topic of the magnetic interactions between the electrons and the nucleus has multiple aspects and is the subject of numerous works, we began here with our first study on the most-known expressions of these interactions, e.g. spin–orbit and spin–spin coupling. Our motivation is based on the fact that these interactions, very weak at the atomic levels, become very strong and even dominant near the nucleus. Thus, they have to be taken into account for the EDOs. To this end, we analyzed some works of Barut, carried out in a relativistic context, and some studies related to the well-known Vigier–Barut model (VB) developed in a non-relativistic context. These works aim to find tight-orbit solutions with high binding energies.

- We noted that Barut found resonances for positronium, for a very short inter-particle distance $\sim$0.02 F and huge energy of order GeV. The author expected the extension of his method (based on a Dirac equation "completed" by the AMM of the electron) to the atom H, but we did not find this extension in the literature, and it seems very difficult on account of the finite dimension of the proton, whereas in positronium the particles are considered "point-like". Nevertheless, in another article of Barut, we found a result on tight (H or D) molecules with high binding energy $\sim$50 keV. The non-relativistic works on the VB model use Schrödinger-like or Pauli-like equations with a potential in the form of a sum of inverse powers of the radial distance. There are few numerical results about deep orbits in the literature, but we noted results for a tight state with energy ~ -40 keV and for tight molecules with – 28 keV (H_2) and – 56 keV (D_2). These results have a size near the result (Section 2) we found with a special iterative process [5] for coherence of the energy w.r.t. the radii.
- Next we presented some computations based on magnetic interactions. As result, we can observe the *HUR, expressed relativistically, can certainly be satisfied by EDOs*, thanks to the high energies of these interactions: in particular the spin–spin coupling is sufficient. Our computations are carried out in a very simplified context and so they lead to an approximate conclusion; nevertheless, it is already a hopeful result.
- Computations were also carried out to look for a local minimum near the nucleus, to provide stability for EDOs. Unfortunately, the local minimum tends to take place inside the nucleus, as the repulsive potential in $1/r^4$ is much too weak to overcome, in the EDO region, the strong attractive potentials we computed. So, we have to continue to work on this question, by taking into account further interactions and effects occurring near the nucleus and by using adequate relativistic quantum tools. In particular, in a QED context, the Two Body Dirac Equation of Dirac's constraint dynamic [41], used in several works by Crater, e.g. [42], allows one to express, in a consistent relativistic way, numerous magnetic interactions at short distances, under the form of "quasi-potentials". Moreover it gets around the Currie–Jordan–Sudarshan "non-interaction theorem" [43], about the bad habit of adding simple potentials to obtain a relativistic Hamiltonian dynamics.
- There is also a further open question: how can we obtain deep orbits from atomic H (or D)? Of course, it seems difficult to believe such a transition could happen spontaneously, for free atoms, e.g. in the spatial vacuum. Nevertheless we can think about the following possible process in condensed matter: from a computer simulation [21] cited in Section 4.3., mini-atoms could be generated from H atoms confined in a metallic net and exposed to a beam of slow quasi-free electrons (from the conduction band) set in motion by low-energy electric arcs. In the simulation, these mini-atoms are unstable; but, when the electrons approach the nucleus, they could be "hooked" by strong magnetic interactions, such as spin–spin coupling, and so they could remain bound. Of course, one has to mathematically confirm this qualitative reasoning, perhaps by scattering computations.

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